Pattern Generalisation in Secondary School Mathematics: Students' Strategies, Justifications and Beliefs and the Influence of Task Features

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DECLARATION

I hereby declare that the work presented in this thesis is my own.

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ABSTRACT

Number pattern generalisation is often regarded a difficult topic for students to learn. To explore this perception, the present study undertakes an empirical investigation with the main aim of providing a comprehensive description of how 14-year-old secondary school students in Singapore generalise figural patterns and justify their generalisations when varying the formats of pattern display and the types of function. Comprising two inter-related parts, the study first examines 515 students' strategies and justifications and probes systematically the influence of the formats of pattern display and the types of function on their generalisations through a specially developed paper-and-pencil test. The other part, through a specially designed questionnaire, looks at their beliefs about which strategy would best help them to derive the rule for predicting any term of a figural pattern as well as their ability to construct the rule using their choice of strategy.

The first part uses an independent-measures research design to examine whether different formats of pattern display have any effect on students' rule construction and a repeated-measures research design to determine whether their rule construction is influenced by the different types of function. In the second part, a survey study is employed with all students asked to identify their choice of best-help generalising strategy. This is then followed by interviews with 16 of the 515 students to probe whether they are able to derive a correct functional rule using their chosen strategy.

This study complements many previous studies mainly undertaken in the west in that its findings indicate that the more academic students are competent in developing a functional rule for linear patterns but falters when working with quadratic patterns. There is a widespread failure of the less academic students in both linear and quadratic patterns, confirming the oft-regarded view that expressing generality is elusive. Successful students perceive the patterns in several ways and generate wide-ranging functional rules, predominantly symbolic, to describe them. They employ a variety of generalising strategies, especially the figural type, and some of which are new in the literature. Both the test and the survey confirm that the figural strategy involving the breaking up of the whole configurations into non-overlapping parts is their clear favourite. For rule justification, verifying it using the numerical cues and drawing diagrams to explain its development are their favourite approaches. Task features such as the format of pattern display and the type of functions do contribute to student difficulties in generalisation. Based on these findings, some useful teaching strategies for teachers and teacher educators are then suggested to help them improve their teaching of pattern generalisation. The findings also point the direction for future research studies on pattern generalisation by suggesting some recommendations for researchers.

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CHAPTER 1 : INTRODUCTION

1.1 BACKGROUND OF THE STUDY

Generalisation is an important aspect of the learning process. It can take place from a very young age when toddlers begin to explore and get acquainted with the world they live in (Mason, 2008). For instance, they show that they are naturally inclined to inducing patterns from objects that they come in contact with in their daily lives when they discover and learn that candy is sweet after having their first taste of it. Besides learning to generalise through experience, generalisation also occurs in other different learning contexts, for instance, in the learning of addition in school. When children notice that 3 + 4 and 4 + 3 have the same sum, and so do 5 + 8 and 8 + 5 as well as 6 + 9 and 9 + 6, they might be able to convey the idea that the order in which two numbers are added is not important. They are said to have made a mathematical generalisation because they are capable of articulating what they have seen in just a few examples to all cases.

Besides arithmetic, generalisation also permeates many other topics in the context of mathematics learning. It can occur in the learning of algebra and geometry, where laws and theorems can be considered as generalisations (Mason, 1996). A case in point for algebra is the commutative law for addition. The numerical identities 3 + 4 = 4 + 3, 5 + 8 = 8 + 5 and 6 + 9 = 9 + 6 can be generalised to a + b = b + a. In the case of geometry, when children are given triangles of various shapes and sizes, and are asked to investigate the sum of the interior angles of a triangle, they may notice that the sum is always 180° regardless of the types (i.e., equilateral, isosceles or scalene) and sizes of triangles. So this observation, although based on just a few particular triangles, may already be sufficient to lead them to make some kind of generalisation about the sum of the interior angles of any triangle being 180°. As the above illustrations make clear, generalisation is an important process in our lives.

Generalisation is thus widely considered a big idea in mathematics, with many researchers going as far as to state that it is at the heart of mathematics (Kaput, 1999; Mason, 1996). This is because generalising is a fundamental and valuable skill central to mathematics learning. My interest in generalisation began some years ago with my curiosity of how children's awareness of generality evolves, how they use their innate powers to see patterns and then make generalisations, and what skills successful children have that unsuccessful children do not that enable them to generalise correctly. Then when I was teaching mathematics at the secondary school level, it was only natural for me to take a special preference for mathematical generalisation, in particular, the kind of generalisation that involves writing a general rule for the n^{th} term of a number pattern. Such pattern generalising tasks are a powerful vehicle not only for introducing the notion of variables (Lee, 1996; Mason, 1996), but also for initiating meaning-making of algebraic expressions (Kieran, 2004) when algebra is used as a mathematical language for expressing generality. Additionally, these tasks help to develop two core aspects of algebraic thinking: the emphasis on relationships among quantities like the inputs and outputs (Radford, 2008) and the idea of expressing an explicit rule using letters to represent numerical values of the outputs (Kaput, 2008). Of particular importance is the latter, which is perceived as a goal of pattern generalising tasks by several researchers (Dreyfus, 1991; Ellis, 2007; Radford, 1996).

This interest in pattern generalisation was further piqued by an issue frequently reported in the GCE "O" level[1] and "N" level examiners' reports about student inability to construct a functional rule for simple linear and quadratic sequences. For instance, the examiners' report for 1996 "O" level examination highlighted that the question asking for the n^{th} term of the sequence [4, 9, 16, 25, …] had relatively few correct answers (University of Cambridge Local Examinations Syndicate, 1997). The examiners were surprised that even the very strong students had difficulty in dealing with it. Whilst a significant number of students could recognise that they were dealing with square numbers, it was noted that most of them gave the popular wrong answer n^2, without realising that the first term was not 1^2 but $(1+1)^2$. Another manifestation of the students' difficulty in expressing generality was revealed in the examiners' report for 2004 examination (Cambridge International

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1 The General Certificate Examination (GCE) is a national examination conducted by the Cambridge International Examinations syndicate in collaboration with the Ministry of Education, Singapore. The GCE at the Ordinary level (GCE "O" Level) and the GCE at the Normal level (GCE "N" Level) are taken by the Express and Normal (Academic) students respectively at the end of their fourth year of secondary education.

Examinations, 2005). The wrong answer $n+3$ was fairly common to the question asking for the n^{th} term of a sequence whose first five terms were given as [1, 4, 7, 10, 13].

Over the years, students' difficulties in deriving the functional rule are still very much in evidence, even in the recent GCE "O" level examinations, despite the teaching and learning of number patterns in Singapore secondary schools since the 1990s. In the 2007 examination, many students realised that n^2 had to be an element of the functional rule for the quadratic sequence [9, 16, 25, 36, …], but could not make the correct adjustment required (Cambridge International Examinations, 2008). The examiners' comment mirrored that for another similar quadratic sequence in the 1996 examination that was described in the last paragraph. Similarly, in the 2009 examination, the generalising question that only supplied the first term 38 and the recursive rule that each following term is found by subtracting 7 from the previous term again stumped many students. When asked to establish an expression for the n^{th} term of the sequence, $38-7n$ was a very common wrong answer given by the majority (Cambridge International Examinations, 2010a).

The recurring comment in the examiners' reports about many Singapore students performing poorly in expressing the functional rule was worrying to me as a mathematics teacher previously and now a mathematics teacher educator. The compelling evidence in those reports drew my attention directly to a cause for concern – students' difficulties with expressing generality. But this concern is not without justifiable grounds because generalising tasks like the four examination questions above have been a feature in the Singapore school mathematics curriculum materials for over a decade, and were not, therefore, totally unfamiliar to students. Yet there is little evidence suggesting that the situation has improved as students are still struggling with such tasks after all these years. However, the difficulties experienced by Singapore students are not surprising as they find strong echoes in the literature. Hence, drawing from the findings of many research studies undertaken in different countries, one crucial conclusion that emerges is that generalisation is elusive for many students and its process is often fraught with difficulties.

Student difficulties have been well documented in the literature. In fact, several studies have revealed that students, even though able to recognise a valid pattern, could not articulate a number pattern in words or translate the pattern in algebraic notation (English &

Warren, 1995; Stacey & MacGregor, 1997, 2001; Ursini, 1991). Other obstacles that students face include the failure to recognise what Lee (1996) called "algebraically useful patterns" (p. 95), the use of wrong generalising strategies to construct the function rules (Lannin, Barker, & Townsend, 2006a; Moss & Beatty, 2006; Stacey, 1989), and the failure to figure out the pattern when it is presented figurally (Becker & Rivera, 2006; Warren, 2005).

These difficulties can be traced to a few possible factors related to the students themselves: inexperience in working with generalising tasks (Stacey, 1989; Warren, 2005), ignorance of appropriate generalising strategies (Moss & Beatty, 2006), lack of spatial visualisation techniques (Becker & Rivera, 2006; Warren, 2005), inexperience in using the highly specific mathematical language of algebra to express generality (Hoyles, Noss, Geraniou, & Mavrikis, 2009) and lack of an understanding of the variable concept (Becker & Rivera, 2006). Whilst student factors such as these are typically perceived as impediments to successful expression of generality, there are now studies suggesting that these factors might not be the main causes of student difficulties, but some might emanate from the tasks themselves.

In some recent studies, student difficulty in expressing generality seemed to be attributed to certain features of the generalising tasks. For instance, the format in which a figural pattern was displayed, whether as a single configuration or as a sequence of configurations, appeared to affect students' choice of generalising strategy (Lannin et al., 2006a). Next, two-dimensional and sequential configurations seemed to help students to better visualise the underpinning pattern in a study by Becker and Rivera (2006) but not in another separate study by Warren (2000). These suggestions were mostly drawn from personal opinions and insights of the researchers rather than from empirical data collected in those studies. Furthermore, the success rates of students in generalising tasks involving just a single configuration are noticed to be usually low (see Cañadas, Castro, & Castro, 2011; Hoyles & Küchemann, 2001). Consider the "classic" matchstick task in TIMSS–2007. A single configuration showing a row of four squares made of 13 matchsticks was provided and Year 8 students were asked about the number of squares in a row that could be made using 73 matchsticks. The success rate for Year 8 students internationally was barely 9%

compared to about 41% for Singapore students (Foy & Olson, 2009). Although Singapore students had outperformed many of their counterparts from other countries in that matchstick task, their success rate was rather low, considering Singapore being a top-performing nation in the TIMSS–2007 study. Further, the low success rate of Year 8 students internationally appears to suggest that expressing generality is a far bigger issue in other countries than it is in Singapore. Since the task features have not been systematically controlled for in those studies, research on their effects on students' expression of generality would seem a promising and worthy research theme.

There is a wealth of research in the literature that identified several kinds of generalising strategies used by students at different grade levels: for example, primary (see Gan & Ghazali, 2007; Lannin, 2005; Warren & Cooper, 2008a), secondary (see Drury, 2007; Lee & Freiman, 2006; Radford, 2006; Rivera & Becker, 2008; Steele, 2008) and even high school (see Dindyal, 2007). In some of these studies, students were even asked to justify how different-looking rules for a pattern could all be equivalent to one another (see Drury, 2007; Lee & Freiman, 2006; Rivera & Becker, 2008). This is an excellent task to challenge the students to come up with multiple ways of seeing the same pattern using the various generalising strategies. However, none of these studies went further to ask students for the kind of generalising strategy that they believe would best help them to express the functional rule. Thus, it would seem worthwhile to survey students' beliefs about which strategy would be the most helpful for rule construction.

In the light of the above discussion, this present study posits that student difficulties in rule construction might be influenced by certain task features, and, therefore, aims to investigate their effects on students' rule construction. To gain a deeper insight into the students' performance in generalising tasks, their generalising strategies, rules and the way their rules are expressed, justifications and beliefs of best-help strategies will be explored.

1.2 THE SINGAPORE CONTEXT

This section presents the Singapore context in which the present study is embedded. It comprises an overview of the Singapore school system at the primary and secondary levels, and the content of number patterns in the primary and lower secondary mathematics syllabi.

Children in Singapore begin their six years of compulsory primary education at the age of seven. At the end of Primary Six, students (usually 12 years old) take the Primary School Leaving Examination (PSLE), which is a national examination, in English, Mother Tongue, Mathematics and Science. Depending on their performance at the PSLE, the students are placed in one of the three courses offered at the secondary level: Express, Normal (Academic) or Normal (Technical) course. These courses, whose medium of instruction is also English, have different curricular emphases that are designed to suit the students' learning abilities and interests. Of the Secondary One cohort, 60% made it to the Express course, 30% for Normal (Academic) course and 10% for the Normal (Technical) course. Students in the Express course are considered to be academically more able than those in the two Normal[2] courses. The distribution of Secondary One students in these three courses is summarised in Table 1.1 below.

Table 1.1: Distribution of PSLE cohort in secondary school courses

Courses	Express	Normal (Academic)	Normal (Technical)
Percentage of Secondary One students	60%	30%	10%

The secondary education is four years for students in the Express and Normal (Technical) courses and five years for those in the Normal (Academic) courses. At the end of the fourth year, Express students take the GCE "O" level examination whilst Normal students take the GCE "N" level examination.

Mathematics is a compulsory subject for all primary and secondary school students in the Singapore education system. Both the primary and secondary mathematics curricula and syllabi are stipulated by the Singapore Ministry of Education. Mathematics textbooks are written in accordance with these syllabi; schools generally adhere to the syllabi closely although they are given leeway to sequence the mathematics topics in any logical order that they deem appropriate for students to learn.

2 The term "Normal", when used without any mention of the word, Academic or Technical, refers to both the Normal (Academic) and Normal (Technical) courses.

According to the secondary mathematics syllabus, pattern generalising tasks are formally introduced in Secondary One across all three courses[3] after students have learnt how to use letters as variables to represent unknown numbers in algebraic expressions and formulae. In learning this topic, students are required not only to recognise and extend number patterns, but also to find an algebraic expression for the n^{th} term as well. This way of introducing the notion of generalisation after developing the concept of variables using letters in the Singapore secondary mathematics syllabus appears to be different from a widely accepted approach advocated by many mathematics educators around the world: that is, to introduce the concept of variables through generalisation (English & Warren, 1995; Lee, 1996; Mason, 1996). Whether such a variation in students' experiences in learning generalisation would really make any difference in their ability to generalise patterns is definitely worth revisiting after analysing the data. Table 1.2 below shows the Secondary One syllabus for algebra in all three courses.

Table 1.2: Secondary One syllabus for algebra

Topic	Content	Express	Normal (Academic)	Normal (Technical)
Algebraic representation and formulae	Using letters to represent numbers	✓	✓	✓
	Interpreting notations: • ab as $a \times b$ • $\frac{a}{b}$ as $a \div b$ • a^2 as $a \times a$; a^3 as $a \times a \times a$; a^2b as $a \times a \times b$ • $3y$ as $y + y + y$ • or $3 \times y$ • $\frac{3 \pm y}{5}$ as $(3 \pm y) \div 5$ or $\frac{1}{5} \times (3 \pm y)$	✓	✓	✓
	Evaluation of algebraic expressions and formulae	✓	✓	✓
	Translation of simple real-world situations into algebraic expressions	✓	✓	✓
	Recognising and representing number patterns (including finding an algebraic	✓	✓	✓

3 Secondary One students are placed in one of the three courses: Express, Normal (Academic) or Normal (Technical) course, in accordance with their performance at the Primary School Leaving Examination (PSLE).

	expression for the n^{th} term)			
Algebraic manipulation	Addition and subtraction of linear algebraic expressions	✓	✓	
	Simplification of linear algebraic expressions, e.g. • $-2(3x-5)+4x$ • $\frac{2x}{3}-\frac{3(x-5)}{2}$	✓	✓	
	Factorisation of linear algebraic expressions of the form • $ax+ay$ (where a is a constant) • $ax+bx+kay+kby$ (where a, b and k are constants)	✓		

[Source: Ministry of Education, Singapore. (2007). *2007 Mathematics (Secondary) Syllabus*. Singapore: Curriculum Planning and Development Division.]

Even though secondary school students are formally taught to deal with generalising tasks in Secondary One, they should be no strangers to such tasks which are an extension of what they have learnt in primary schools. At the lower primary level, simple generalising tasks like those illustrated in Figure 1.1 are common. These numerical generalising tasks typically require them to study the pattern and then find the missing numbers. At the upper primary level, figural generalising tasks such as the second example in Figure 1.2 may be given in addition to the usual numerical type. The difficulty level of these tasks has increased in that students are expected not only to find the missing terms in the pattern, but also to use inductive reasoning to identify both the recursive and functional rules that define the pattern. The rule is, however, usually kept simple to the linear type of the form $N = an$ or $N = n + b$, and the strategy used is often the method of differencing.

Lower Primary (aged 7 to 9 years)
Fill in the missing numbers. 1. 8, ____, 6, 5, 4 2. 1248, ____, 1268, ____, 1288, 1298

Figure 1.1. Lower primary generalising tasks

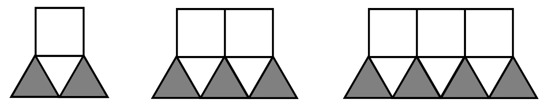
Upper Primary (aged 10 to 12 years)

1. Fill in the missing numbers. Write down the rule for the set of numbers.

   78 560, 77 560, ____, ____, 74 560, 73 560

   Rule: ________________ to get the next number.

2. Study the pattern and answer the following questions.



(a) Fill in the table.

Number of white squares	Number of grey triangles
1	2
2	3
3	4
4	
20	
N	

(b) Describe the pattern you see.

The number of grey triangles is ___________ *the number of white squares.*

(c) How many white squares are there if there are 50 grey triangles?

Figure 1.2. Upper primary generalising tasks

When students reach Secondary One, they are encouraged to build on their previous experience in generalising patterns to solve the numerical as well as figural generalising tasks. Apart from typical questions like generating certain terms in the sequence when given their positions, what is new to most students will be the construction of the functional rule for predicting the n^{th} term in the form of an algebraic expression. In addition, the tasks given at this level will also become more demanding as well, now involving rules that are both linear, usually of the form $N = an + b$, and quadratic, albeit not as common as the linear type. Therefore, generalising tasks encountered at the secondary level are far more challenging than those given at the primary level. Figures 1.3 and 1.4 illustrate some typical generalising tasks offered in the Singapore secondary mathematics textbooks.

To sum up, Singapore is so high achieving in international studies, yet, in the field of pattern generalisation, very little is known about the kind of generalising strategies Singapore secondary school students use to establish the functional rule underpinning a pattern, the kind of rules they construct, the way they express that rule, and the way they justify how they derive that rule. In short, their generalisations and justifications still remain largely unexplored so far and looking at this promising research area is therefore of great interest. Consequently, this present study seeks to illuminate the students' performance in this area.

1. Write down the next two terms of each sequence.
   (a) 11, 13, 15, 17, …
   (b) 1, 2, 4, 8, …
   (c) 1, 3, 6, 10, …

2. Consider the sequence 4, 10, 16, 22, … .
   (a) Find its general term.
   (b) Hence find its 25^{th} term.

Figure 1.3. Secondary numerical linear generalising tasks

[Source: Discovering Mathematics 1B, Ng, Y. C. E. (Ed), pp. 55 – 58. Publisher: Star Publishing Pte Ltd.]

The following shows a sequence of figures made with cubes.

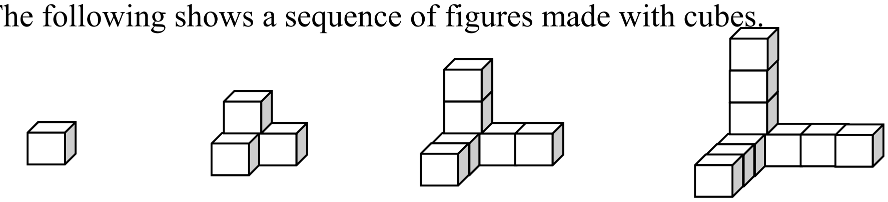


Figure 1 Figure 2 Figure 3 Figure 4

(a) Determine a possible pattern and draw the next figure according to that pattern.
(b) Find the number of cubes in the 10^{th} figure.
(c) Find the number of cubes in the n^{th} figure.

Figure 1.4. Secondary figural linear generalising tasks

[Source: New Express Mathematics 1, Lee, P. Y. & Fan, L. H. (Eds), p. 127. Publisher: Multimedia Communications]

1.3 AIMS OF THE PRESENT STUDY

The aims of the study are briefly outlined below:

(a) to examine how secondary school students make and justify generalisations of number patterns when certain task features are varied;

(b) to probe systematically the effect of certain task features on students' generalisations; and

(c) to highlight what students believe to be the most helpful generalising strategies.

1.4 MAIN RESEARCH QUESTIONS

The present study was primarily concerned with Singapore secondary school students' generalisations of figural patterns, justifications of how they made that generalisation, beliefs of the most helpful generalising strategy and the influence of task features on their rule construction. It aimed to investigate the following four main research questions:

1. How do Singapore secondary school students establish the rule that defines a figural pattern?
2. How do Singapore secondary school students justify the rule they constructed?
3. How do task features influence Singapore secondary school students' rule construction?
4. What do Singapore secondary school students judge to be the most helpful generalising strategy for constructing the functional rule?

In order to answer these questions, this study was devised and conducted in two inter-related parts: Study I and Study II. Study I, called *an investigation of students' generalisation and justification of patterns, and the effect of task features on their rule construction*, adopted an experimental design to address the first three research questions. This first part investigated the way Singapore secondary school students established and justified the functional rule underpinning a pattern and the influence of task features. Study II was a survey-cum-student interview study, called *an exploration of what students believe is the best-help generalising strategy*. To address the fourth research question, this second part examined the kind of generalising strategy that students judged as the most helpful for rule construction and explored the efficacy of their choice of best-help strategy on rule construction. To seek empirical answers to the four research questions, students were asked to complete both a paper-and-pencil test and a questionnaire, from which their rules, generalising strategies, justification schemes, as well as the way they expressed the rules and their choices of generalising strategies that they judged as most helpful were determined.

Figure 1.5 presents the general research framework of the research. It depicts the four variables involved in the study: generalisation of pattern, task features, justification and student belief. The relationships amongst them are illustrated by the arrows. The elaborations of these four variables are given in the detailed research framework in Section 2.8. Based on this detailed research framework, the four main research questions are further extended to specific ones, which are presented in Section 2.8 as well.

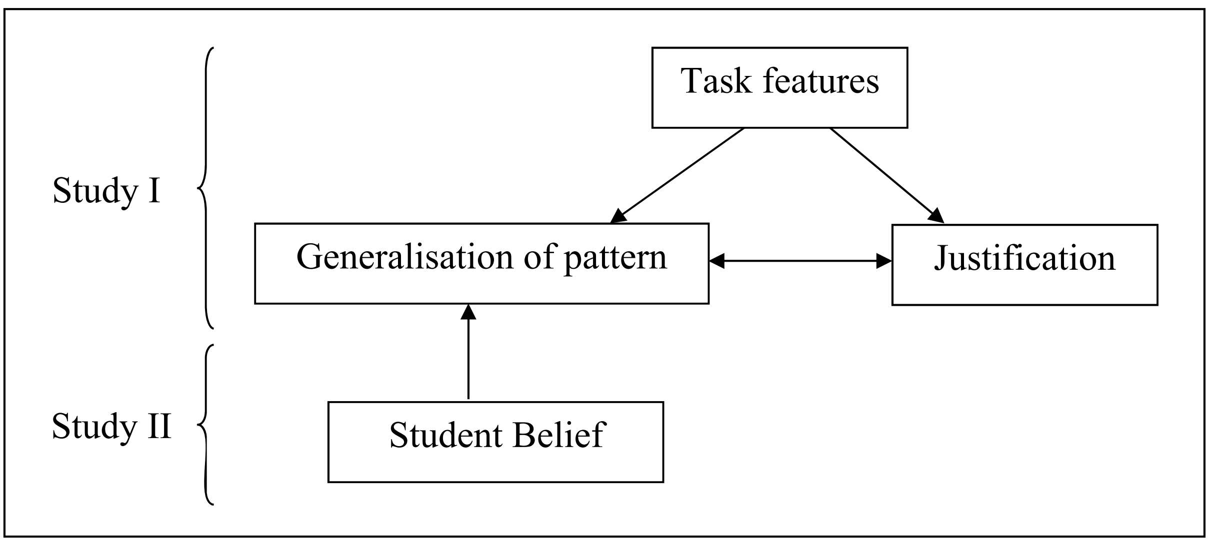


Figure 1.5. General research framework

As the general research framework shows, Study I and Study II are related to each other, with generalisation of patterns being a common theme throughout the studies. The findings of Study I will provide a comprehensive picture of Singapore secondary school students' generalisations and justifications. As indicated in Section 1.1, generalisation remains notoriously elusive for many students, who often fail to correctly establish the functional rule. There is thus a need to ascertain the kind of generalising strategy that could help the students to articulate the functional rule successfully. The findings of Study II will help to illuminate the helpful generalising strategies.

1.5 DEFINITIONS OF KEY TERMS

This section seeks to define the eight key terms as used in the present study: generalisation, generalising task, task features, the functional rule, generalising strategy, the modality of the rule, the efficacy of a generalising strategy, and justification.

(a) The term *generalisation* has been extensively discussed in the literature by several researchers (see Dreyfus, 1991; Dubinsky, 1991; Ellis, 2007; Harel & Tall, 1991; Kaput, 1999; Radford, 1996). For instance, Ellis (2007) defines it as a process which engages learners in at least one of the following three activities:

   i. identifying a commonality across cases,
   ii. extending one's reasoning beyond the range in which it originated, or
   iii. deriving broader results from particular cases (p. 197).

   In the present study, Ellis' operational definition of generalisation is adopted because of its applicability to generalising tasks involving number patterns.

(b) A *generalising task* is a kind of mathematical problem which can be solved by "examining specific cases, organising the results systematically, finding a pattern and using it to get the answer" (Stacey, 1989, p. 147). All the examples in Figures 1.1 to 1.4 qualify as generalising tasks. Although such tasks are also labelled as *generalising problems* by Lee and Wheeler (1987) in the literature, this study decides not to adopt their phrase. This is for two reasons. First, the word *problem* suggests a situation of which students are aware but do not know how to proceed directly to solve. Although research has shown that this might be the case for many students, the mathematical problem in question might not be a challenging problem for those who succeed. In contrast, the word *task* seems to be more general, clearly conveying the idea of a piece of work that must be solved. Second, the term *generalising task* also corresponds nicely with another key term used in this study: *task feature*.

(c) The term *task features* refers to the defining characteristics that constitute the entire generalising task (Chua, 2009). The two task features being investigated in this study are described below.

   i. **The format of pattern display** In figural generalising tasks, diagrammatic configurations are used to depict the pattern. This task feature is concerned with whether the pattern is presented as a sequence of successive or non-successive diagrammatic configurations.

ii. **The type of functions** The underpinning pattern in a generalising task can be represented as a relationship between two variables. This task feature considers whether this relationship describes a linear or non-linear rule. In this study, the non-linear relationship employed is of the quadratic type.

(d) The term *functional rule*, which has been used by many researchers such as Hunter and Anthony (2008), Küchemann (2010), and Moss and Beatty (2006) in the literature, refers to the position–to–term relationship, expressed as a function, that defines the pattern depicted in a generalising task. The rule is useful for calculating immediately any term of the pattern when its position is known.

(e) A *generalising strategy* refers to the approach encompassing a series of steps taken to interpret and describe a pattern structure depicted in a generalising task.

(f) The *modality* of a functional rule refers to the way the rule is expressed. The three common ways of expressing the rule are:

(i) using words (e.g., add three to twice the size number),

(ii) using mathematical symbols (e.g, $2n + 3$),

(iii) using a combination of both words and mathematical symbols (e.g., $2 \times size\ number + 3$).

(g) The *efficacy* of a generalising strategy is judged by whether the strategy helps students to derive the correct functional rule.

(h) A mathematical justification can take on different meanings, depending on the context it occurs in (see Balacheff, 1988; Hoyles & Healy, 1999; Simon & Blume, 1996). In the case of pattern generalisation, a *justification* is taken as an explanation given in response to the question asking how a rule underpinning the pattern in a generalising task is constructed (Becker & Rivera, 2009; Lannin, 2005). This way of defining the term requires the students to illuminate the generalising strategies used to obtain their rules.

1.6 STRUCTURE OF THE REPORT

This chapter outlines the background and research approach adopted in this study that guides towards the accomplishment of the goals of this study. As the above discussion has clearly shown, expressing generality remains a huge challenge for numerous students in many countries, even for those with strong mathematical attainment. However, Singapore secondary school students' generalisation and justification strategies are still not widely understood and thus remains an intriguing area to explore. But, before this study can proceed further, it is necessary to review past research that has bearing on the research. Therefore, a critical literature review is presented in the next chapter and its conclusion is used to frame the research questions for this study. This second chapter ends with a detailed elaboration of the research framework and a presentation of the specific research questions guiding the study. Following the literature review, descriptions of the research design, the participating schools and students, the test instruments, the procedures of both the pilot and main studies, the scoring rubric and data coding schemes, as well as the data analysis plan to be carried out on the collected data, are detailed in Chapter 3. In Chapter 4, the findings of Study I, collected from both quantitative and qualitative analyses of data, are reported in great detail and used to address the first three research questions. Chapter 5 reports the findings of Study II, similarly drawn from both quantitative and qualitative data, and the findings are subsequently used to address the fourth research question. Both Chapters 4 and 5 conclude with a summary of the major research findings in the respective parts of the present study. Finally, Chapter 6 discusses the key findings presented in Chapters 4 and 5 in the light of past research in the literature and, drawing from this discussion, follows with a conclusion of these key findings. The significance and limitations of this study, the implications for teaching, learning and curriculum design, as well as suggestions for further research are also presented in this chapter.

CHAPTER 2 : LITERATURE REVIEW

The topic of number patterns has been included in the Singapore secondary mathematics curriculum for over a decade, yet there is little, if any, local research looking at students' generalisations of patterns. Consequently, the current state of the secondary school students' generalisations of patterns is still not well understood and studied. So the present study undertakes to investigate how the students generalise patterns and justify their generalisations.

To shape the direction and design of the present study, a review of the research literature that bears on the formulation and justification of the expressions of generality was carried out. The first section elaborates what the generalisation process entails as well as the different types of pattern generalisation that have been identified. The next section describes what pattern generalising tasks test and the different types of generalising tasks available in the research literature. Since one of the aims of this study is to investigate the kind of generalising strategies the students use to develop a functional rule underpinning the pattern, it was judged to be essential to discuss the various generalising strategies identified in the literature. The examination of students' generalisation of patterns would not be complete without considering the types of rules the students established and the modality of the rules. It is thus crucial to gain some insight from the current literature on these two aspects of pattern generalisation. Next, since the present study is to be implemented on Singapore students, their performance in pattern generalising tasks in their national examinations and in international studies is examined and reported. Findings from research studies on students' pattern generalisations carried out across the globe are also presented to provide an overview of the international students' performance in pattern generalisation. Following this is a discussion of the role of justification in pattern generalisation and the different justification schemes that students employ to explain their generalisations. Finally, the research framework and questions based on the conclusion of the literature review are framed for this study.

2.1 GENERALISATION

As highlighted in the previous chapter, generalising is well known to be a fundamental and valuable skill of mathematics learning. It has many applications in mathematics and appears in many forms. Mason (1996) described it as "the heartbeat of mathematics" (p.65). With such considerable importance in mathematics, what is generalisation through the lens of mathematics educators and what is expected of it? This section attempts to examine these questions.

A review of the research literature throws up different interpretations of the term *generalisation*. On closer examination, these interpretations reveal the dual meaning of the term, denoting on the one hand a *process* and on the other hand a *product* of this process. The two meanings are elaborated next. Following that is a description of the various types of generalisation identified by mathematics educators.

2.1.1 GENERALISATION AS A PROCESS

Several researchers have viewed generalisation as a process of extending one's existing argument or schema beyond the case or cases considered. For instance, Harel and Tall (1991) described it as applying "a given argument in a broader context" (p. 38), and Dubinsky (1991) considered it as acts of extending "an existing schema to a wider collection of phenomena" (p. 101). According to Radford (1996) who examined number patterns, generalisation is a procedure that draws a conclusion α from a sequence of "observed facts", a_1, a_2, …, a_n. This procedure can be denoted symbolically as a_1, a_2, …, a_n → α to mean α is derived from a_1, a_2, …, a_n. In order to interpret the "observed facts", a_1, a_2, …, a_n, learners must be able to conceptualise the mathematical objects and the relations involved in these facts. Radford's definition captures Mason's (1996) sense of "seeing the general through the particular" (p. 65) but neglects certain activities that typically precede the extension of several cases to a general conclusion. These activities include examining a few given cases presented in the task to search for a pattern, recognising the pattern from the given cases and predicting other cases. It therefore appears that Radford's definition of generalisation is limited and needs to be broadened to encompass the aforementioned activities.

Some researchers address the limitations of the existing definitions of *generalisation* by having the neglected activities incorporated into their definitions of the term. One such researcher is Kaput (1999), who described the generalisation process as “deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situation themselves but rather on the patterns, procedures, structures, and the relationship across and among them” (p. 136). His view parallels Dreyfus’ (1991), which not only regarded generalisation as a process requiring learners to “derive or induce from particulars, to identify commonality, to expand domains of validity” (p. 35), but also underscored the importance of establishing “a result for a large class of cases” (p. 35). Ellis (2007) summed up the two interpretations of generalisation succinctly. According to her, generalisation is a dynamic rather than static process that engages learners in at least one of the following three activities:

i. identifying a commonality across cases,
ii. extending one’s reasoning beyond the range in which it originated, or
iii. deriving broader results from particular cases (p. 197).

An interesting point emerging from Ellis’ interpretation of generalisation is her view of what is to be taken as evidence of generalisation. Normally, the formulation of a conjecture, be it verbal or symbolic, is deemed as an appropriate evidence – a point well recognised by some researchers such as Stacey and MacGregor (1997). However, departing from this typical view, Ellis claimed that the evidence is the similarities and extensions learners identify and perceive as general. Clearly, understanding what learners see as common properties among the cases is valued very much. Her viewpoint fits very well with Mason’s (1996) oft-cited phrase of “seeing a generality through the particular” (p. 65), as well as what Radford (2006) sees as a crucial characteristic of generalisation – the “capability of noticing something general in the particular” (p. 5).

Finally, Rivera and Becker (2011) described pattern generalisation as “both actions of constructing an algebraic generalisation and justifying it on the basis of the students’ repertoire of available explanatory mechanisms" (p. 329). Interestingly, their definition manifests two different meanings of the term *generalisation*. By referring to pattern generalisation as actions of rule construction and justification, they viewed generalisation as a process. However, treating the rule as a generalisation manifests the other meaning of the term *generalisation*, which is discussed next.

2.1.2 GENERALISATION AS A PRODUCT

When the formal rule derived from a generalising task is referred to as a generalisation, the term *generalisation* is viewed as a product. Becker and Rivera (2005, 2006) are two researchers who have adopted this interpretation, as evidenced in their definition of pattern generalisation provided above and the following research questions they posed in two of their studies:

1a) What strategies do successful students use to develop an explicit generalisation?

b) How do students make use of visual and numerical cues in developing a generalisation? (Becker & Rivera, 2005, p. 122)

2 How do sixth graders acquire the ability to establish and justify generalisations in algebra? (Becker & Rivera, 2006, p. 465)

Sharing the same view are Kaput, Blanton and Moreno (2008) who referred to generalisation as "a single statement that applies to multiple instances without making a repetitive statement about each instances … [by means of referring] to multiple instances through some sort of unifying expression that refers to all of them in some unitary way, in a single statement" (p. 20).

2.1.3 TYPES OF GENERALISATION

Much of the research examining students' generalisations have accumulated rich and valuable information about their strategies in expressing generality, thereby offering researchers many opportunities to examine the different types of generalisation produced. What follows in the rest of this section is a discussion of the different types of generalisations that the researchers have established.

Several types of generalisations have been determined, but not all are pertinent to pattern generalisation. For instance, Harel and Tall (1991) identified three different types of generalisation that depend on the cognitive activities involved in making generalisations: (1) *expansive*, which is based on expanding the applicability range of an existing scheme, without reconstructing the schema; (2) *reconstructive*, where the existing schema is reconstructed in order to widen the applicability range; and (3) *disjunctive*, where a new schema is constructed when moving to a new context. As the descriptions make clear, these generalisation types tend to be of general application to any other mathematical activities, not just particularly to pattern generalising tasks.

Researchers such as Radford, Rivera and Becker have investigated the way students create expressions of generality for patterns for many years and documented a few types of pattern generalisations. Radford (2008) distinguished between *arithmetic* and *algebraic* pattern generalisations. An *arithmetic* pattern generalisation occurs when students engage in empirical counting to identify the commonality and generalise it to the rest of the terms without providing a direct rule for finding any particular term. It often happens when students employ the popular recursive strategy to extend the pattern through using the common difference between successive terms. Although recognising the recursive relationship between successive terms helps to extend the pattern, it does not, however, predict any term immediately when given its position in the pattern. In contrast, an *algebraic* pattern generalisation rests on "the capability of grasping a commonality noticed on some particulars (say $p_1, p_2, p_3, \ldots, p_k$); extending or generalising this commonality to all subsequent terms ($p_{k+1}, p_{k+2}, p_{k+3}, \ldots$), and being able to use the commonality to provide a direct expression of any term of the sequence" (Radford, 2008, p. 84). In other words, the noticed pattern structure is abstracted, or objectified in Radford's language, into an expressed generality. By engaging in structure recognition, the algebraic generalisation is therefore considered more sophisticated than the arithmetic type, which fails to evoke the "algebraic nature" (Radford, 2008, p. 85) of the former type despite manifesting a generalisation.

Algebraic generalisations are further classified into the following three types: : factual, contextual and symbolic (Radford, 2006). A *factual generalisation* occurs when the generality is found through forming a scheme that operates on just numbers. The expression of the generality remains confined to a numerical level and the explicit rule remains unnamed. In a *contextual generalisation*, the generality is established by naming the general objects through a description that uses linguistic and non-symbolic terms: for instance, using "the next figure" to refer to the contextual objects. Unlike factual generalisation, the indeterminate in the contextual generalisation is now made explicit. A *symbolic generalisation* is one that uses alphanumeric symbols to express the generality, in which the indeterminate is also linguistically explicit, just like in a contextual generalisation. Amongst these three types of generalisation, the symbolic ones are located at the deepest layer of generality whilst the factual ones are at the topmost layer.

Radford distinguished between an arithmetic and algebraic pattern generalisation, maintaining that algebraic thinking was "certainly neither about guessing nor about just using signs" (Radford, 2006, p. 15) but about "thinking in certain distinctive ways" (Radford, 2001, p. 84). Thus a generality expressed symbolically through trial and error does not foster algebraic thinking when there is no involvement of any recognition of pattern structure. Therefore, what really characterises a generalisation distinctively algebraic is the reasoning and structure discernment involved in formulating an expressed generality rather than the use of algebraic notation.

Whilst Radford categorised algebraic generalisations according to two attributes: (1) the different layers of generality characterised by semiotic cues such as words and gestures, and (2) the corresponding mode of expression such as notations, Rivera and Becker (2008) classified algebraic generalisations for figural patterns according to the students' generalising strategies. They identified two basic types of generalisation: constructive generalisation and deconstructive generalisation. A *constructive* generalisation occurs when the configurations used to depict a pattern are broken up into non-overlapping parts and a formal rule is established by adding up these parts. When different parts are overlapped to form the configurations and the rule is expressed by counting separately each part of the configuration then followed by subtracting the overlapping parts, a *deconstructive* generalisation is produced.

2.2 PATTERN GENERALISING TASKS

Pattern generalising tasks are a common feature of school mathematics in many countries including, for instance, Australia, Portugal, Singapore, the United States of America (USA), and the United Kingdom (UK). They are mathematical tasks typically concerned with:

(a) examining a few cases of numbers or figures presented in the task to search for a pattern,

(b) recognising the pattern from the given cases,

(c) continuing the pattern to predict the next couple of cases,

(d) articulating the pattern in words, in diagrams or frequently in symbolic forms,

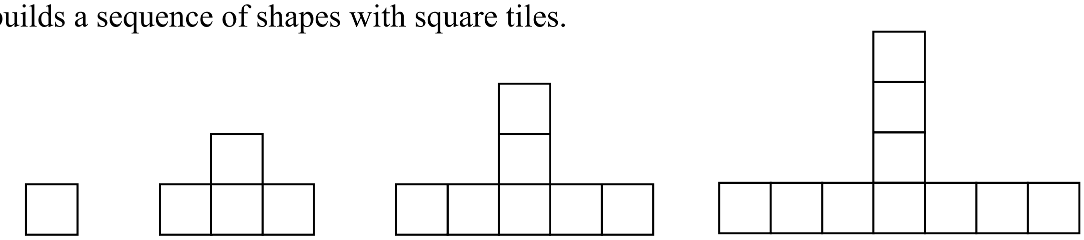
(e) calculating specified cases using the newly established pattern.

By and large, pattern generalising tasks can be classified into two categories: *numerical* and *figural*. Numerical generalising tasks list the pattern as a sequence of numbers whereas figural generalising tasks set the pattern in a pictorial context. In other words, figural patterns are visual representations of numerical patterns. The two kinds of generalising tasks in Figure 2.1 illustrate the visual-numeric relationship between them, with the figural task in (b) underpinning the same numerical pattern as the one shown in (a). Both tasks, often used by mathematics teachers in class to illustrate pattern generalisation, are known as growth patterns because of a tacit assumption that the pattern in each task grows predictably from one term to the next in the sequence of numbers or figures (Billings, 2008).

> The first five terms of a sequence are given as follows:
>
> 1, 4, 7, 10, 13, …
>
> (a) Write down the 6^{th}, 10^{th} and 50^{th} terms.
> (b) Can you write down a rule for finding the n^{th} term in this sequence if you were told what n is? Show how you obtained your answer.

(a) Numerical generalising task

Ken builds a sequence of shapes with square tiles.



Shape 1 Shape 2 Shape 3 Shape 4

(a) Write down a rule for finding the number of tiles needed to build shape N. Show how you obtained your answer.
(b) Using the rule you obtained in (a), find the number of tiles needed to build Shape 100.
(c) Using the rule you obtained in (a), find which shape is built from 178 tiles.

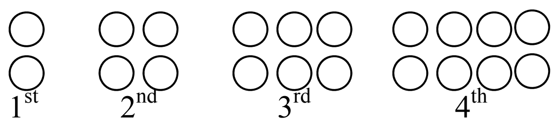
(b) Figural generalising task

Figure 2.1. Pattern generalising tasks

In part (a) of the task presented in Figure 2.1(a), finding the sixth term requires students to examine the five given terms and infer from these terms the pattern that each subsequent term is three more than the previous term, then finally apply this pattern to predict the next term that comes after 13. Finding the 10^{th} and 50^{th} terms asks for what Stacey (1989) calls a near and far generalisation of the pattern respectively. The term *near generalisation* is used to denote finding a term that is not immediately next to the given terms but yet near enough by step-by-step counting whereas *far generalisation* denotes finding a term that is very much further away from the given terms in which step-by-step counting is no longer a practical approach for finding it. However, in Figure 2.1(b), students do not have to make any far generalisations in answering parts (b) and (c); the answers for the specific cases can be directly deduced from the rule established in part (a).

The two tasks in Figure 2.2 below, studied by Healy and Hoyles (1999), as well as Warren and Cooper (2008a), are also classified as growth pattern, but they differ from those in Figure 2.1 in the way the patterns grow. Each pattern in Figure 2.3 contains an identifiable core: a column of two counters ( ) in (a), and a pair of black and white tiles ( ) in (b). The constant growth in each pattern is a result of cyclically repeating the identifiable core, unlike the non-cyclical growth in Figure 2.1(b). In consideration of the way the pattern grows, these tasks are occasionally referred to as repeating patterns in the literature. To derive an expression for the nth configuration in these two tasks, as well as the one in Figure 2.1(b), students have to rely on the independent (that is, the figure number) and dependent (that is, the number of tiles or chips) values, and then search for an invariant relationship between them before abstracting this relationship as a rule.

Ken builds a sequence of shapes with chips.



Can you help Ken find the number of chips needed to build the n^{th} shape?

When there are 3 black tiles, there are 3 white tiles.

When there are 7 black tiles, there are 7 white tiles.

Can you find the number of white tiles if you know the number of black tiles?

(a) The counter arrays task (b) The lines of tiles task

Figure 2.2. Examples of repeating patterns

As the examples clearly illustrated, the skills behind generalising tasks that students are expected to develop are identifying a numerical pattern, extending the pattern to make a near and far generalisation, and articulating the functional relationship that describes the pattern using symbols. Numerous researchers maintain that such tasks are a powerful and useful vehicle for promoting and supporting algebraic thinking. The three merits of exploring such generalising tasks include:

(a) they can be used to introduce the notion of a variable (Mason, 1996; Warren & Cooper, 2008b),

(b) they can be used to develop two core aspects of algebraic thinking: (i) the emphasis on relationships among quantities such as the inputs and outputs (Radford, 2008), and (ii) the idea of expressing an explicit rule using letters to represent numerical values of the outputs (Kaput, 2008; Kieran, 1989), and

(c) they can be used to develop the notion of equivalence of algebraic expressions (Warren & Cooper, 2008b).

With such merits that are pivotal to fostering algebraic thinking, it is not difficult to understand why pattern generalisation is taught normally under algebra in many countries, including the US and Singapore.

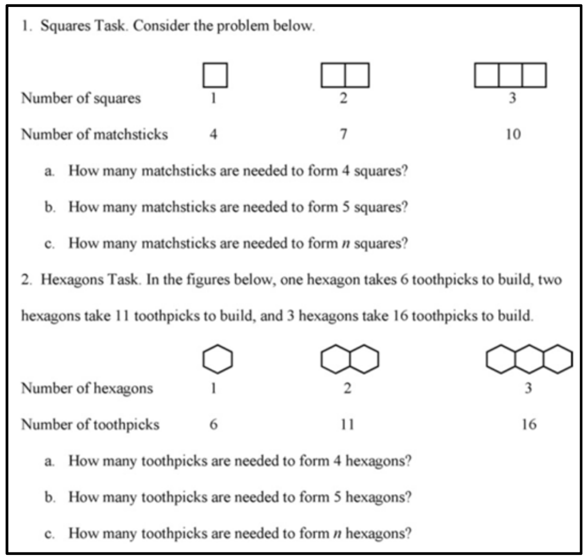
Certain generalising tasks, in particular the numerical type, do have a pitfall. A rigourous mathematical analysis of the numerical pattern reveals its inherent ambiguous nature. The pattern can grow in numerous ways, yet rarely does the generalising task describe precisely how the pattern should grow. As a result, the generalising task is vague and it is possible to produce many equally viable and valid rules to define the numerical pattern (Rivera and Becker, 2007). According to Whitton (1971), a student could write down *any* number when asked to supply the next term of a sequence. Using the numerical sequence 1, 4, 7, 10, 13, … provided in Figure 2.1(a) as an illustration of Whitton's argument, the expression $3n-2$ is a rule for the general term of the sequence but, for any value of k, so is the expression $[3n-2+k(n-1)(n-2)(n-3)(n-4)(n-5)]$. Clearly, k can be computed so as to make any chosen number become the next term of the sequence. For instance, if the sixth term is chosen to be the number 14, substituting n by 6 into the expression $[3n-2+k(n-1)(n-2)(n-3)(n-4)(n-5)]$, followed by equating the result with 14 yields the value of $-\frac{1}{60}$ for k. Therefore, the sequence, if extended, becomes 1, 4, 7, 10, 13, 14, …, $\left[3n-2-\frac{1}{60}(n-1)(n-2)(n-3)(n-4)(n-5)\right]$, … . Whitton is not wrong to infer the rule for the general term in this manner, but the problem is the technique of formulating the correct rule runs the risk of becoming algebraically too complex for secondary school students to follow through and write down. For the sake of simplicity, the rule formulation is typically accomplished by having the students to make tacit assumption that the pattern grows in a predictable manner from one term to the next (Billings, 2008). This means that the incremental change from one term to the next of a linear pattern remains constant. As for a quadratic pattern, the incremental change varies by the same amount. Unlike numerical patterns, figural patterns seem to have eluded such a pitfall because of the explicit nature of the configurations by which the patterns are defined.

In the corpus of research focussing on pattern generalisation, there is a wide range of pattern generalising tasks in terms of the way the patterns are presented, as well as the type of functions involved. These two features of generalising tasks are discussed below.

2.2.1 FORMAT OF PATTERN DISPLAY

Numerical generalising tasks tend to have a rather consistent format of pattern display, typically listing at least the first four terms of a pattern in a sequential order. For instance, students in Stacey's (1989) study were given the first four terms of a linear sequence [4, 10, 16, 22, …] and asked to find only the 100th term in the sequence. The same pattern format was also spotted in another number sequence, [3, 6, 11, 18, …], used in a separate study (Stalo, Elia, Gagatsis, Theoklitou, & Savva, 2006). But in the study of Hargreaves, Threfall, Frobisher and Shorrocks-Taylor (1999), every generalising task that was administered to students listed the first five terms of the number patterns: for instance, the linear pattern [2, 5, 8, 11, 14, …] and the quadratic pattern [2, 4, 7, 11, 16, …].

In contrast, figural generalising tasks tend to show more variations in the format of pattern display. A very widely used approach is to provide three or more successive configurations to represent the figural pattern. The literature comprises several examples of such figural tasks (see Billings, 2008; Radford, 2008; Rivera & Becker, 2007; Stacey, 1989; Warren & Cooper, 2008a). Figure 2.3 presents two tasks with three successive configurations by Rivera and Becker (2007) in (a), a task with three successive configurations by Billings (2008) in (b), and a task with four successive configurations by Warren and Cooper (2008a) in (c). The figural patterns in (a) and (c) each begin with the first configuration in the respective patterns whereas the pattern in (b) starts with the second configuration instead of the first.



1. Squares Task. Consider the problem below.

Number of squares	1	2	3
Number of matchsticks	4	7	10

a. How many matchsticks are needed to form 4 squares?

b. How many matchsticks are needed to form 5 squares?

c. How many matchsticks are needed to form *n* squares?

2. Hexagons Task. In the figures below, one hexagon takes 6 toothpicks to build, two hexagons take 11 toothpicks to build, and 3 hexagons take 16 toothpicks to build.

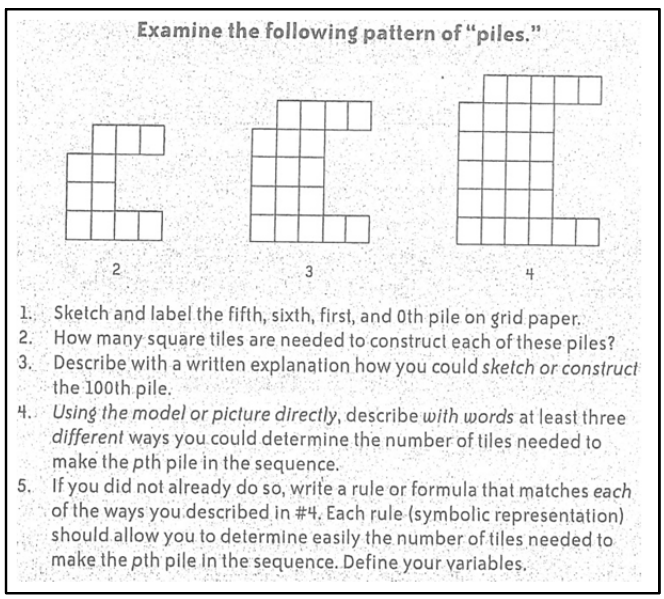
Number of hexagons	1	2	3
Number of toothpicks	6	11	16

a. How many toothpicks are needed to form 4 hexagons?

b. How many toothpicks are needed to form 5 hexagons?

c. How many toothpicks are needed to form *n* hexagons?

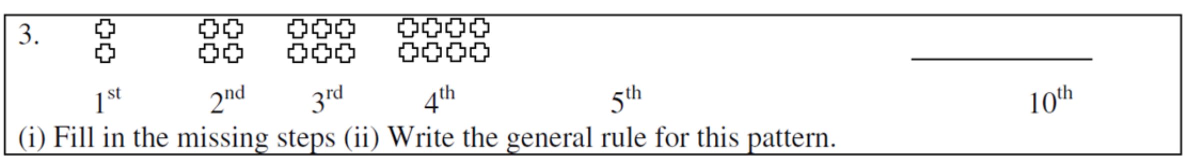
(a) *Squares* and *Hexagons* by Rivera and Becker (2007)



Examine the following pattern of "piles."

1. Sketch and label the fifth, sixth, first, and 0th pile on grid paper.
2. How many square tiles are needed to construct each of these piles?
3. Describe with a written explanation how you could *sketch or construct* the 100th pile.
4. *Using the model or picture directly*, describe *with words* at least three *different* ways you could determine the number of tiles needed to make the *p*th pile in the sequence.
5. If you did not already do so, write a rule or formula that matches *each* of the ways you described in #4. Each rule (symbolic representation) should allow you to determine easily the number of tiles needed to make the *p*th pile in the sequence. Define your variables.

(b) *Piles* by Billings (2008)



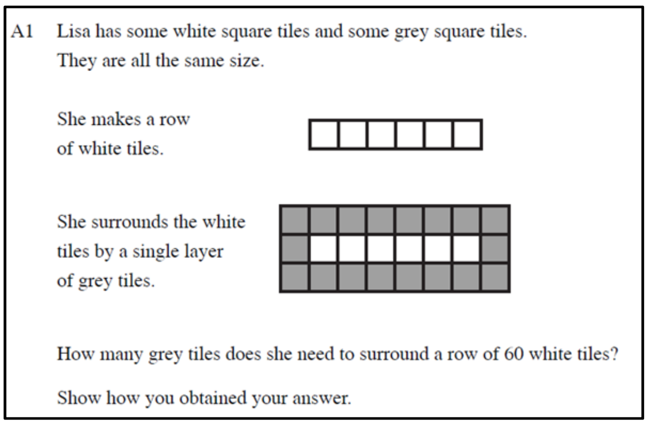
3.

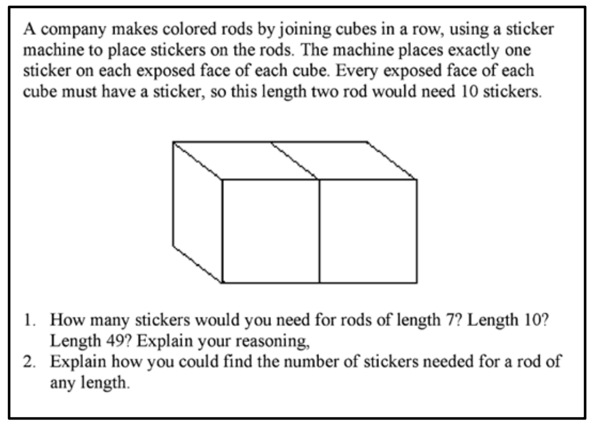
(i) Fill in the missing steps (ii) Write the general rule for this pattern.

(c) Task by Warren and Cooper (2008)

Figure 2.3. Figural generalising tasks with successive configurations

Another widespread approach is to show just a single configuration to represent a generic case of the figural pattern. Researchers who had explored such tasks in their studies include, for instance, Cañadas, Castro and Castro (2011), Hoyles and Küchemann (2001), Lannin (2005), and Steele (2008). Figure 2.4 presents two figural generalising tasks that provide a single configuration, with the border-tiling task in (a) taken from the study of Hoyles and Küchemann and the *Cube Sticker* task in (b) from Lannin's study. The popularity of the border-tiling task is well established in the literature. Besides Hoyles and Küchemann, other researchers who had used it, or its variations, in their research include Billings (2008), Cañadas, Castro and Castro (2011), Smith, Hillen and Catania (2007), Steele and Johanning (2004), and Taplin and Robertson (1997).





(a) Border-tiling (b) Cube sticker

Figure 2.4. Figural tasks with single configuration

Apart from the above two approaches featuring one or three successive configurations, other less common approaches include providing two or three non-successive configurations. A comprehensive literature review has so far found generalising tasks with two configurations in only five studies. Figure 2.5 below shows a prime example investigated by Warren and Cooper (2008a).

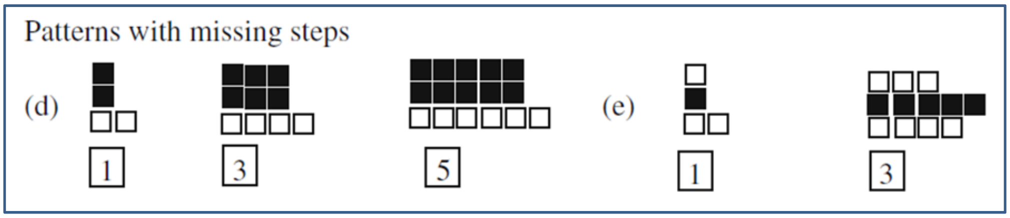
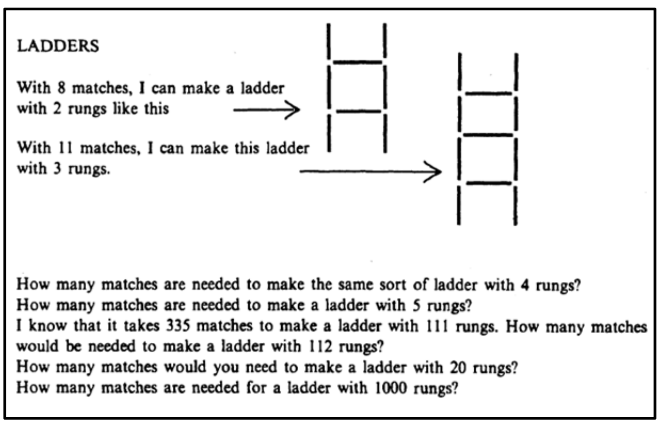
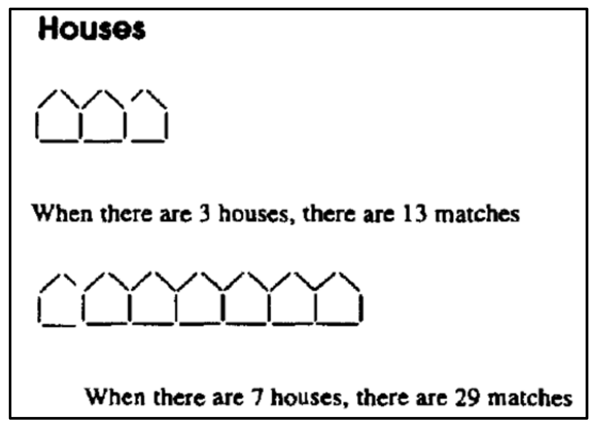


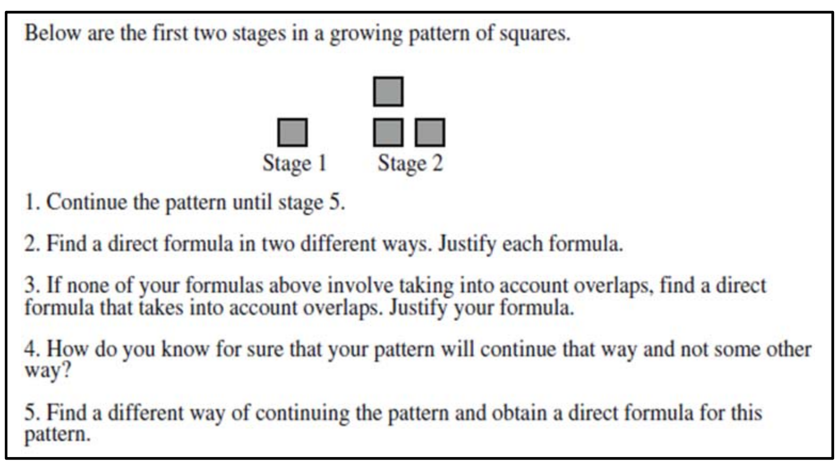
Figure 2.5. Generalising tasks with non-successive configurations

Three other examples, *Ladders*, *Houses* and *Square Pattern* are provided in Figure 2.6 below. The Ladders task from Stacey's (1989) study and the Square P*attern* task from Rivera's (2010) study showed two successive configurations whereas the *Houses* task by Healy and Hoyles (1999), as well as part (e) in Figure 2.5, featured two non-successive configurations. Such tasks are valuable for encouraging students to focus on the position–to–term relationship (Warren & Cooper, 2008a).





(a) Ladders (b) Houses



(b) Square pattern

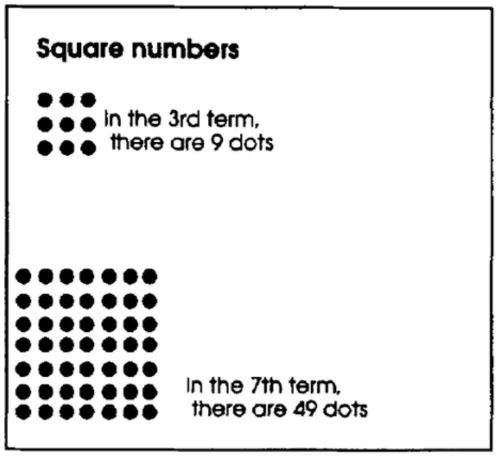
Figure 2.6. Figural tasks with two configurations

2.2.2 TYPES OF FUNCTIONS

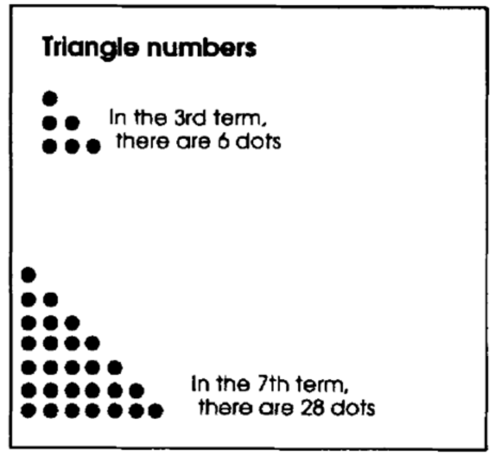
Previous work on pattern generalisation has tended to focus on both linear and quadratic patterns, with linear patterns being more pervasive in the research literature (see Lannin, Barker, & Townsend, 2006b; Radford, 2001; Rivera & Becker, 2008; Stacey, 1989; Warren, 2005). A typical linear pattern is defined by a functional rule of the closed form $an + b$, where n is the independent variable. This rule is said to be in one variable because the output depends on only the input n, which is normally the figure number. Examples of generalising tasks with this form of linear rule include the *Squares* and *Hexagons* tasks in Figure 2.3(a), as well as the *Ladders* and *Houses* tasks in Figure 2.6 above. To illustrate an example, consider the *Hexagons* task. The number of matchsticks required to form a row of n hexagons is represented by the linear expression $5n + 1$.

Besides taking the form of a one-variable expression, the functional rule of a linear pattern can also be expressed in two variables. A well-known example is the pond-tiling task, which is another guise of the classic border-tiling task, mentioned by Noss, Hoyles, Mavrikis, Geraniou, Gutierrez-Santos and Pearce (Noss et al., 2009). The task involves a pond of any given length and width to be surrounded with square tiles. The number of tiles needed depends on the number of tiles along the length, l, and the width, w, and can be represented symbolically by $2l + 2w + 4$. This type of task is not as commonly investigated as the one-variable linear pattern in research studies.

The next type of generalising tasks involves quadratic patterns, which have been investigated by researchers including, for instance, Billings (2008), Hargreaves et al. (1999), Healy and Hoyles (1999), and Rivera and Leung (2012). Three examples of quadratic tasks are presented in Figure 2.3(b) and Figure 2.7.



(a) Square numbers

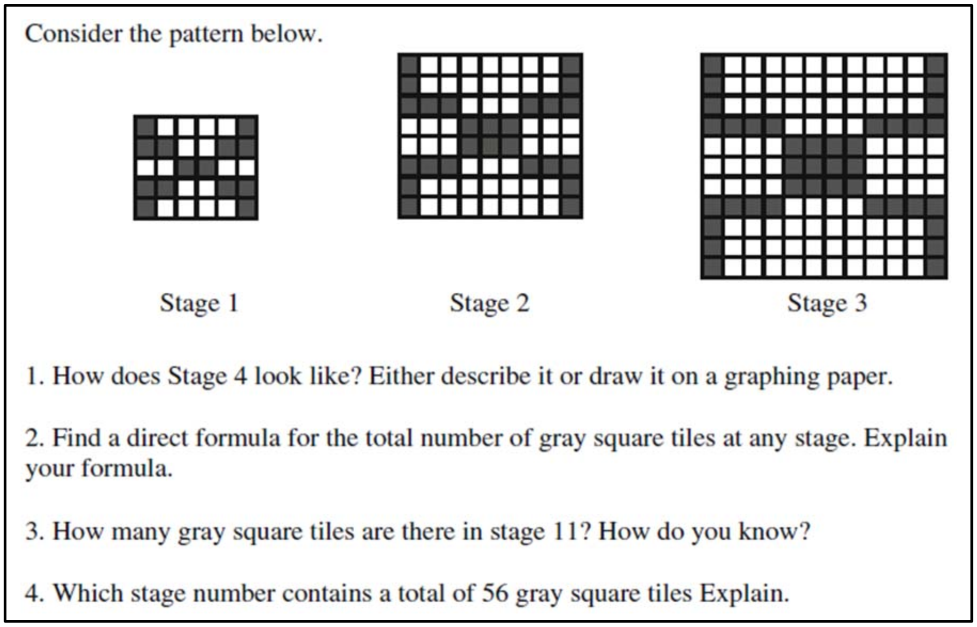


(b) Triangle numbers

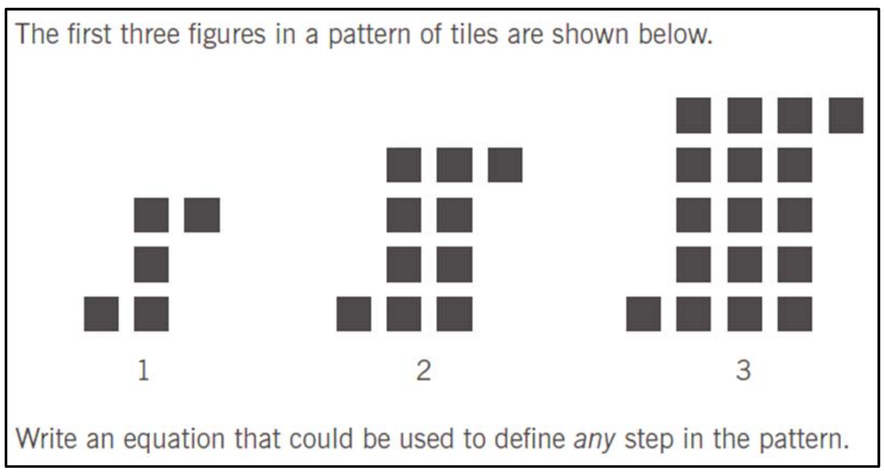
Figure 2.7. Quadratic generalising tasks

The quadratic tasks in the literature typically follow two widely recognised patterns: *square* numbers [1, 4, 9, 16, 25, …] and *triangle* numbers [1, 3, 6, 10, 15, …]. The two generalising tasks in Figure 2.7, taken from the study of Healy and Hoyles (1999), exhibit these patterns. Other manifestations of the two patterns had also appeared in the research of Rivera and Leung (2012) and Steele (2008) as well.

Apart from those two popular patterns, there are other quadratic generalising tasks whose patterns do not conform to any of them. *Piles* in Figure 2.3(b) above, as well as *Square Frog* and *S* in Figure 2.8 below, are three examples of such generalising tasks. A quadratic expression for finding the number of grey tiles in the n^{th} term of the *Square Frog* pattern, designed by Rivera (2010), is $n(n+1)+4(2n+1)$, or n^2+9n+4 when simplified. For the S task addressed by Smith, Hillen and Catania (2007), the quadratic expression $n^2+2(n+1)$ can define any step, n, in the pattern.



(a) Square Frog



(b) S

Figure 2.8. More quadratic generalising tasks

2.3 TYPES OF GENERALISING STRATEGIES

The structure of a pattern is interpretable in many different ways. A comprehensive review of the body of work on students' generalising strategies and reasoning has highlighted a range of generalising strategies that students used to envision and interpret the pattern structure. Some of the generalising strategies are prevalently used in making far generalisations and others in formulating the functional rules. A discussion of the various types of generalising strategies follows in the next two sub-sections.

2.3.1 GENERALISING STRATEGIES FOR FINDING SPECIFIC TERMS

Stacey (1989) was one of the first to examine student use of generalising strategies for calculating values of particular terms in linear generalising tasks. She identified three strategies commonly used by students, namely

(a) *Counting* Students draw a picture to represent the situation and then count the desired attribute.

(b) *Difference* Students build a larger unit by multiplying the unit's position by the common difference. Taking the *Squares* task in Figure 2.3(a), for instance, the difference in matchsticks between any two consecutive rows of squares is three. So to calculate the number of matchsticks needed to form five squares, students evaluate 5×3.

(c) *Whole–object* Students use a known value as a unit to construct a larger unit using multiples of the unit. For instance, a row of two squares in the *Squares* task requires seven matchsticks, so a row of 10 squares requires five times as many matchsticks (that is, 5×7).

Stacey's framework had since been further developed by Healy and Hoyles (1999), as well as Lannin, Barker and Townsend (2006a). Healy and Hoyles introduced another generalising strategy called *interterm chunking between terms*, which requires students to apply the recursive rule to add a unit onto a known value. Using the *Squares* task with a common difference of three to illustrate, if a row of three squares requires 10 matchsticks, a row of five squares can then be formed by adding two units of three matchsticks. Therefore, the number of matchsticks required to form five squares can be expressed as $10 + 2(3)$. Lannin et al. renamed this latter strategy as *chunking* and their framework comprised the three strategies: *counting*, *chunking* and *whole–object*.

As Stacey's (1989) study clearly revealed, the *difference* and *whole–object* strategies do not always apply to all linear generalising tasks and they often lead to erroneous calculations. In particular, the *whole–object* strategy, which involves the incorrect use of proportional reasoning to establish a functional relationship, indicates a serious flaw in student thinking. The errors, therefore, reflect the students' lack of a deep understanding of the direct proportion and linearity concepts. In other words, they did not understand that the two strategies will only work in linear patterns whose terms are directly proportional to their positions. The functional rules of such linear patterns correspond to the form an and *not* $an + b$.

Students using the *chunking* strategy do make mistakes as well (Lannin et al., 2006a). They might, for instance, add the fifth output value (16 in the matchsticks problem) to the tenth output value (31) to find the fifteenth output value (thus finding a solution of 47 instead of 46).

Apart from the above-mentioned strategies, another popular generalising strategy is the *recursive* approach where subsequent terms are built from the previous terms in the pattern. This strategy, included in the framework developed by Lannin *et al*. (2005; 2006b), was used by some of their students to work out *Beam Design* and *Cube Sticker* tasks.

2.3.2 GENERALISING STRATEGIES FOR FINDING FUNCTIONAL RULES

Rivera and Becker (2008) identified two broad categories of strategies that students employed to derive a functional rule, namely

(a) *Numerical*	Students use only cues established from any pattern when listed as a sequence of numbers or tabulated in a table.
(b) *Figural*	Students exploit visual cues established directly from the structure of configurations used to depict the pattern.

Between the two strategies, the numerical approach appears to be more versatile because of its applicability in both numerical and figural generalising tasks. In contrast, the figural approach is relevant only when the pattern is depicted pictorially. But it can offer strong support for the algebraic representation of figural patterns (Healy & Hoyles, 1999). In some studies (see Chua & Hoyles, 2009; Rivera & Becker, 2008), students were found using a combination of both the *numerical* and *figural* strategies in rule construction.

Different types of *numerical* strategy have been described. Bezuszka and Kenney (2008) identified three that involve recursion: (1) *comparison*, where the terms in a given number sequence are compared with corresponding terms of another sequence whose rule is already known, (2) *repeated substitution*, where each subsequent term in a number sequence is expressed in terms of the immediate term preceding it, and (3) *the method of differences*, where the terms in a sequence and their consecutive differences are compared and equated with corresponding terms and differences of a generic expression of the explicit formula that is a polynomial equation.

To illustrate how each *numerical* strategy works, consider the *Squares* task presented in Figure 2.3(a). The first configuration shows a square made of four matchsticks, the second a row of two squares made of seven matchsticks and the third a row of three squares made of 10 matchsticks. So converting the figural pattern into a numerical pattern and with an extension of two more terms, the sequence [4, 7, 10, 13, 16, …] is obtained.

(a) *Comparison strategy* The original sequence, which follows the "*add three to the previous term*" rule, is compared with the sequence of multiples of three (the common difference) as follows:

Original sequence	4,	7,	10,	13,	16, …
Multiples of three	3,	6,	9,	12,	15, …

It is clear that each term in the original sequence is one more than the corresponding term of the sequence formed by the multiples of three. Since the functional rule describing the latter sequence is $3n$, an expression for the n^{th} term of the original sequence is, therefore, $3n + 1$ in notations.

This strategy requires students to be familiar with some key sequences and their functional rules, and to have some ability in translating the difference between the corresponding terms of the two sequences into an algebraic expression.

(b) *Repeated substitution strategy* Starting with the first term, 4, of the original sequence, the second term is one *three* more than the first, the third is two *threes* more than the first, the fourth is three *threes* more than the first, and so on. Thus each subsequent term of the sequence can be expressed as the number of *threes* to be added to the first term 4, and this number of *threes* is one less than the term's position number in the sequence. Hence, this way of interpreting the pattern structure leads to constructing the functional rule $4 + 3(n - 1)$.

(c) *Method of differences strategy* The original sequence underpins a linear rule since its first difference is constant. So the terms and common difference of the original sequence are then compared with those of a generic linear sequence defined by $pn + q$, as shown in Figure 2.9.
Comparing the first differences, it follows that $p = 3$. After identifying the value of p, the value of q is found to be 1 by equating and solving $p + q = 4$ where $p = 3$. So the functional rule of the original sequence is $3n + 1$.

n	Original sequence	First difference
1	4	
		3
2	7	
		3
3	10	
		3
4	13	

n	$pn + q$	First difference
1	$p + q$	
		p
2	$2p + q$	
		p
3	$3p + q$	
		p
4	$4p + q$	

Figure 2.9. Method of differences strategy

Amongst the three *numerical* strategies described here, the *method of differences* strategy is most useful for finding a quadratic functional rule. The *comparison* and *repeated substitution* strategies are commonly used to construct linear functional rules. Taking, for instance, the *Squares* task in Figure 2.3(a), some participants spotted the common difference, 3, of the linear sequence, wrote down $3n$, and noticed that each term in the sequence was "always 1 more than 3 times *n*" (Rivera & Becker, 2007, p. 149). So the rule $3n + 1$ was developed using the *comparison* strategy. One participant in the same study noticed that each subsequent term of the sequence could be expressed as the number of *threes* to be added to the first term 4, and next, and that this number of *threes* was always one less than the term's position number in the sequence. Following this observation, the *repeated substitution* strategy was then used to work out the rule $4 + 3(n - 1)$.

Some students might have been taught how to work out a linear function rule using the formula $a + (n - 1)d$, as evidenced in the 2010 GCE "N" level examiners' report for Singapore students (Cambridge International Examinations, 2011). They have to identify the first term, *a*, of a sequence and the common difference, *d*, between any two consecutive terms first, and then substitute these two values into the formula. The formula represents a generic expression for the n^{th} term of any linear sequence using the *repeated substitution* strategy. The "formula" strategy is, however, not the same as the *repeated substitution* strategy if the derivation of the formula is not explained clearly to the students. The lack of explanation often results in students merely knowing how to obtain the formula mechanically without developing a good grasp of why it works.

Similarly, different categories of *figural* strategy have also been identified. Rivera and Becker (2008) distinguished between *constructive* and *deconstructive* strategies. The former occurs when the configuration given in a generalising task is viewed as a composite configuration made up of non-overlapping components and the rule is directly expressed as a sum of the various sub-components. The latter happens when the configuration is visualised as being made up of components that overlap, and the rule is expressed by separately counting each component of the configuration and then subtracting any overlapping parts.

A problem with the generalising strategy framework developed by Rivera and Becker (2008) is that it does not account for two other strategies. One of them involves rearranging one or more components of the original configuration to form something more familiar. The newly-rearranged configuration highlights the structure of the pattern which then facilitates the rule construction. The other entails viewing the original configuration as part of a larger composite configuration, from which the rule is generated by subtracting the sub-components from this composite configuration. Therefore, Chua and Hoyles (2010a) introduced these two strategies into the existing framework by Rivera and Becker (2008). The first one was called the *reconstructive* strategy and the second was known as *figure–ground reversal*.

In order to illuminate the figural strategies described above, four illustrations using the *S* task (see Figure 2.8) are provided in Figure 2.10, with each showing how the structure of the *S* pattern can be interpreted using a different type of the figural strategies. The *S* configuration in (a) is separated into three non-overlapping blocks: a horizontal block each at the top and at the bottom, plus a square block in the middle. This way of interpreting the *S* configuration uses the *constructive* strategy. In (b), the *deconstructive* strategy is employed when two identical “thinner” *S* shapes are overlapped to form the Step-2 configuration. The tiles that overlap are circled in the diagram. Another way to envision the pattern structure is to use the *reconstructive* strategy, which is shown in (c). The top row of tiles is shifted, then rotated 90° and finally re-positioned on the left of the original configuration. The resulting configuration subsequently reveals the pattern structure. Finally, in (d), the *S* configuration can be imagined as part of a larger square initially, but with two identical columns of tiles removed subsequently. This last illustration exemplifies the use of the *figure–ground reversal* strategy.

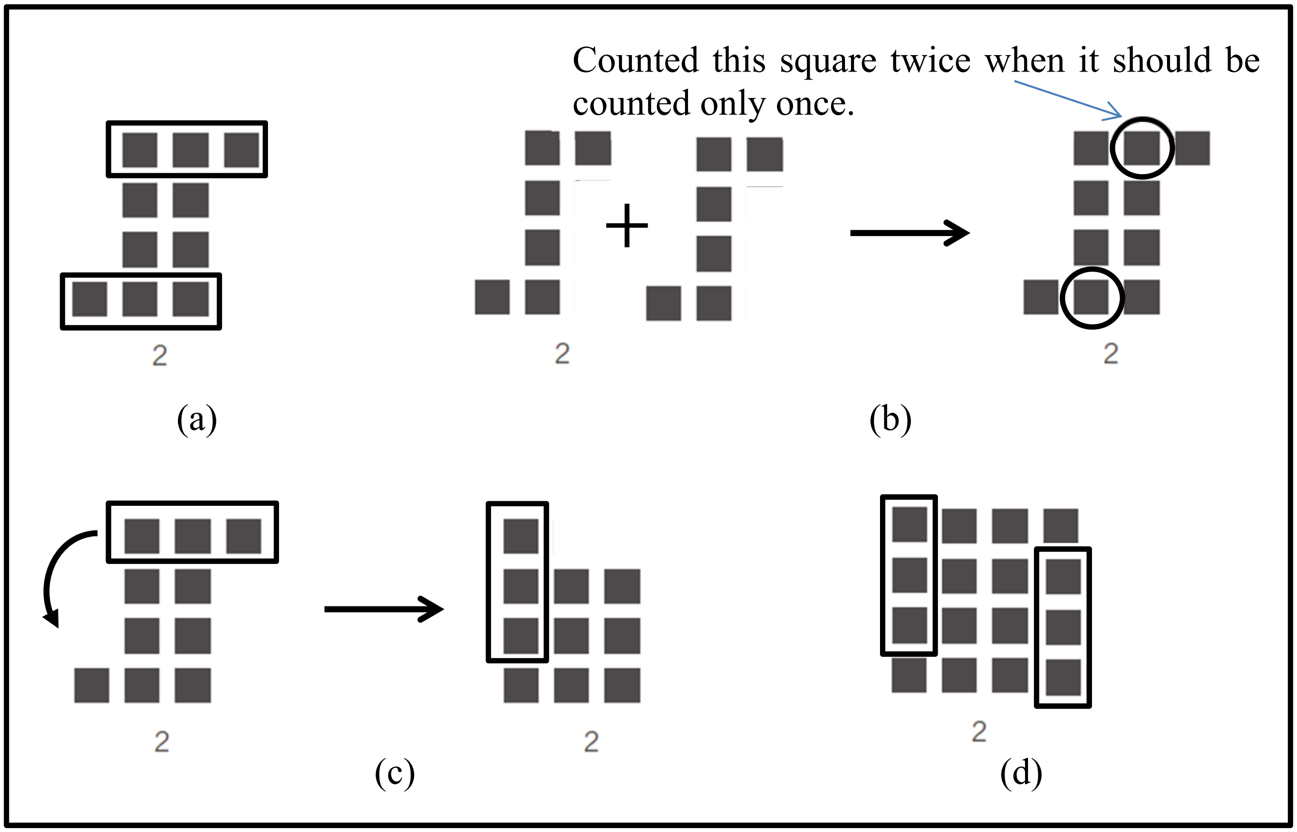


Figure 2.10. The S task using different figural strategies

An interesting observation to emerge from the literature review is that of the four figural strategies, the *constructive* strategy appears to be a clear favourite amongst both young and older learners, judging from its frequent occurrence in research studies (see Moss & Beatty, 2006; Radford, 2006; Rivera, 2010; Steele, 2007). There were also some cases of students' use of the *deconstructive* strategy (see Becker & Rivera, 2007; Lannin, 2005; Steele & Johanning, 2004). However, Rivera and Becker (2008) had observed that this strategy was not as commonly known and adopted as the *constructive* strategy amongst the students in their study. Further, it also appears that only older students from Year 7 and above are familiar with such a strategy.

The use of the *reconstructive* and *figure-ground reversal* strategies is relatively infrequent in research studies. Cooper and Warren (2011) reported a Year 5 Australian student's use of the *figure-ground reversal* strategy to generate a functional rule for a two-column tower of blocks, with one of the columns missing a block at the top. Initially this student imagined a full two-column tower of blocks, then subtracted one from the total number of blocks to indicate the action of taking away the extra block that was added at first. The same strategy was also engaged by a Year 8 US student in Rivera's (2010) study when each L-shaped configuration in the figural pattern was viewed as being cut out from a larger square. Another US student in the same Rivera's study employed a combination of *reconstructive* and *figure-ground reversal* strategies to the L-shaped pattern to work out a functional rule.

Clearly, the use of a combination of different generalising strategies in rule construction is possible. Taking, for instance, Figure 2.10(c), the resulting configuration can be envisioned as a single square resting on a 3 by 3 square and so the n^{th} term is given by the expression $(n+1)^2+1$. This way of viewing the pattern structure engages both the *reconstructive* and *constructive* strategies. Alternatively, when the resulting configuration is interpreted as a 4 by 3 rectangular array missing two squares, the expression for the n^{th} term is $(n+2)(n+1)-n$. In this latter illustration, the two generalising strategies involved are *reconstructive* and *figure-ground reversal*, which are identical to the aforementioned US student's in Rivera's study. Unfortunately, such combinations of strategies are rarely established in the research literature on pattern generalisation. So far, apart from combining *reconstructive* and *figure-ground reversal* strategies, Becker and Rivera (2006) had reported on another type of combination, which they termed as a *pragmatic* generalisation, involving both the numerical and figural strategies.

Apart from those generalising strategies described above, the *guess-and-check* strategy is employed by some students in rule construction (Moss & Beatty, 2006; Radford, 2006; Rivera & Becker, 2007). In this approach, students usually test different algebraic expressions through experimenting with various parameters such as operations and numbers in the expressions and some terms of the pattern in the generalising tasks until they find one that fits those few terms under consideration. The *guess-and-check* strategy often leads to making attempts to find a rule to fit a particular instance of the pattern rather than understanding what the parameters represent in relation to the pattern in the task (Becker & Rivera, 2005; Lannin, 2005). This explains why amongst those who use such a strategy to produce a correct rule, there is not always an understanding of the position–to–term relationship when asked to explain why the rule might work.

2.4 TYPES OF RULES

Students are often required to construct a rule to describe the pattern structure that they see in a generalising task. From the review of several studies in the research literature (see Healy & Hoyles, 1999; Lannin et al., 2006b; Radford, 2006; Rivera & Becker, 2007; Stacey, 1989), it is observed that the rules constructed by the students take on mainly two forms: *recursive* and *functional*. In the literature, the phrase *recursive rule* is generally understood to be used to refer to the rule that allows the computation of the next term of a sequence using the immediate term preceding it. On the other hand, the functional rule computes the term directly using its position in the sequence. Researchers have used somewhat different terminology for naming such a rule in the literature. It is sometimes known as an explicit rule or formula (see Becker & Rivera, 2005; Lannin et al., 2006b), a general rule or formula (see Rivera & Becker, 2008; Warren & Cooper, 2008a), or a closed-form formula (see Küchemann, 2010; Moss & Beatty, 2006). These terms are used interchangeably by most researchers. Throughout this thesis, the phrase *functional rule* is used. Using the *Squares* task above, an example of the *recursive* rule is *add three to the previous term* whereas the *functional* rule can be represented by $3n + 1$, $4 + 3(n - 1)$, $2n + (n + 1)$ or $4n - (n - 1)$, depending on which generalising strategy was used to visualise the structure of the squares.

According to Lannin et al. (2006b), a recursive rule involves “recognising and using the change from term–to–term in the dependent variable” (p. 300) whilst a functional rule uses “reasoning that relates the independent variable to the dependent variable(s)” (p. 300). Students tend to produce a recursive rule when dealing with generalising tasks that “provided a clear connection to incremental change” (Lannin et al., 2006a, p.12), which happens when “the input values were relatively close” (Lannin et al., 2006a, p.12) in the tasks. This finding seems to be sensible because by building on the previous terms in a pattern, the recursive approach allows subsequent terms to be determined effortlessly. As a result, such an approach is particularly popular amongst many students, especially the younger ones (Hargreaves et al., 1999). Unfortunately, the recursive rule, whilst useful for finding subsequent terms by first knowing the preceding terms, suffers from some serious drawbacks. First, it is not efficient for the immediate calculation of any term whose position number is a large value or when the pattern is presented in a non-successive manner. Next, it does not promote the ability to examine the functional relationship between the terms and their positions, a viewpoint which many researchers have argued is key to algebraic thinking (Kaput, 2008; Mason, Graham, & Johnston-Wilder, 2005; Radford, 2008). Finally, articulating correctly the recursive rule of a quadratic pattern in words is by no means a straightforward task (Hargreaves et al., 1999). This is why formulating a functional rule is deemed so crucial and helpful to students.

2.4.1 DIFFERENT EQUIVALENT FORMS OF RULES

As mentioned in the previous section, the four expressions $3n + 1$, $4 + 3(n - 1)$, $2n + (n + 1)$ and $4n - (n - 1)$, are the functional rules for the *Squares* task in Figure 2.3(a). In particular, the last three expressions, albeit structurally distinct looking, are the different equivalent forms of $3n + 1$ before simplification. These different-looking rules often emerge as a result of different student reasoning, interpretation and discernment of the pattern structure. For instance, the rule, $2n + (n + 1)$, is constructed when a student visualises the row of squares as being composed of two horizontal rows of n matchsticks and $(n + 1)$ vertical matchsticks. If the student builds the row of squares by considering the number of matchsticks (i.e., $(n - 1)$ sets of three) to be added to the first configuration of four matchsticks, then the rule is $4 + 3(n - 1)$. As the two illustrations exemplify, the form of the rule produced reflects the thinking and structure discernment of the student. The two expressions, $2n + (n + 1)$ and $4 + 3(n - 1)$, despite taking different forms, are equivalent in the sense that both describe the same pattern. But both differ in that they provide two completely different interpretations of the same pattern. So as Arcavi (1994) puts it, although the two expressions are equivalent, they actually convey completely non-equivalent meanings.

The idea of discussing equivalence of expressions with students had been proposed much earlier by Arcavi (1994) but, in recent years, it had garnered more interest from several researchers such as Becker and Rivera (2006) and Moss, Beatty, Barklin and Shillolo (2008). This growing interest reflects the researchers' desire in encouraging students to become versatile and think flexibly about ways to see the pattern structure from multiple perspectives. Amongst the studies that have explored the use of equivalent rules is that conducted by Becker and Rivera (2006), where students had to develop functional rules in different ways and to explain how some given rules might have come about. Encouraging students to construct rules in different ways fosters *perceptual agility*, which Lee (1996) described as the ability to see a pattern in multiple ways, and a willingness to abandon those that do not lead to a functional rule. Moreover, asking students to explain equivalent rules paves the path for establishing the invariance of different rules, regardless of the way one perceives the pattern structure.

2.4.2 MODALITY OF WRITTEN RULES

The external representations of a mathematical idea are conveyed in many ways. For instance, the relationship between two quantities can be represented by using concrete materials, drawing a graph, writing in words, and using symbols. These different modes of representation are referred to as the *modality* of the mathematical idea. For generalising tasks, one of the important roles is to foster algebraic thinking through using letters to express the rules underpinning the patterns (Kaput, 2008). In line with this role, some researchers introduced letters into their generalising tasks to ask specifically for the symbolic representation of the underlying rule. For instance, Rivera and Becker (2007) asked students to find the number of matchsticks needed to make n squares in *Squares* (see Figure 2.3(a)), and Billings (2008) asked for the number of tiles needed to make the pth configuration in *Piles* (see Figure 2.3(b)). Stacey and MacGregor (2001) were two other researchers who required Years 7 to 10 students in their studies to express the functional rules using algebraic symbols. They observed that only a small proportion of students produced correct symbolic rules in two linear tasks. However, expressing a symbolic rule is sometimes not required, especially at the lower level of study when algebra has not been taught. So the next paragraph describes another mode in which a functional rule can be represented.

Some generalising tasks do not make clear the expected modality of the functional rule. Referring to *Cube Sticker* (see Figure 2.4(b)), *Square Frog* (see Figure 2.8(a)) and *S* (see figure 2.8(b)) for instance, these tasks asked for a rule that could be used to determine any configuration in the respective patterns. But none of them states clearly that a symbolic rule is expected. Furthermore, there are students who might not yet know well enough how to use letters to describe the pattern structure in symbolic terms, or, especially those younger ones at the primary level who might not have been introduced to algebra formally. Thus one may then cogently assume that a rule, if written wholly in words, is equally acceptable. In fact, such a practice of "*use[ing] English, a natural language, to describe relationships that are more frequently expressed with algebraic formalisms*" (Bastable & Schifter, 2008, p. 175) is widespread amongst school children. For instance, Stacey and MacGregor (2001) reported that nearly half of their sample of 2000 Australian students in Years 7 to 10 described the functional relationship underpinning a pattern in words. The same researchers underscored the importance of the "verbal description phase" in the "process of recognising a function and expressing it algebraically" (p. 150).

A student who established a functional rule using the alphanumeric form (that is, a combination of words and symbols) was noted in a study by Mavrikis, Noss, Hoyles and Geraniou (2012). The generalising task showed rectangular tables joined at the longer sides forming a longer table, with a chair placed on each shorter side of the tables. At each end of the long table, there are two chairs. When asked to find a general rule for predicting the number of chairs needed to fit around any number of tables, the student produced a correct expression, $2 \times model\ number + 4$, using words instead of a letter to denote the figure number. The rule displays the form of a symbolic rule despite involving both words and symbols.

Drawing from the illustrations above, it is gathered that different modalities of a functional rule may arise out of three circumstances: (1) the students' prior experience, (2) their cognitive level, and (3) the context in which they are asked to generate the rule. Given these circumstances, it appears necessary to allow the modality of the rule to take any of the three forms: written in symbols, written in words, or written in alphanumeric form.

In a similar vein, when a recursive strategy is used to generalise a pattern, the rule underpinning the pattern can be represented in two modes. One way is to describe it in words and the other is to express it in symbolic terms. Considering *Squares* in Figure 2.3(a), for instance, the recursive rule can be *add three to the previous term to get the next* in words or $T_n = T_{n-1} + 3$ in symbols.

2.5 STUDENTS' PERFORMANCE IN GENERALISING TASKS

A clear message following a review of research studies on students' generalisations of patterns conducted across the globe is that whilst students can often recognise a valid underlying pattern and determine particular cases of the pattern, what they find considerably challenging is the articulation of the rule in algebraic notations. Details of student performance in the various studies are described in the two sections below. The first section discusses those studies undertaken largely in the west. The second section begins with a review of the performance of Year 10 Singapore students in the GCE "O" Level and "N" Level examinations and follows with a discussion of Year 8 students' performance in TIMSS studies.

2.5.1 PERFORMANCE OF STUDENTS IN OTHER COUNTRIES

Becker and Rivera (2005) studied the generalisations of different batches of Grades 8 and 9 US students over five years and noticed a similar pattern in their performance. Whilst over 70% of the students in each batch had consistently shown success in computing particular cases of both numerical and figural linear patterns, they experienced significant difficulty when it came to establishing an algebraic rule for the patterns. In a separate study the researchers conducted on 22 Grade 9 students between 2002 and 2003, 13 students reportedly could not even generalise a single linear pattern. Only five students achieved success in all the patterns. To determine both near and far cases of the patterns and to generate the respective rules, the students used a range of generalising strategies, including *numerical*, *figural* and *guess–and–check*. But their strategies appeared to remain predominantly numerical in nature. Steele (2008) found similar results in a study she conducted on eight Grade 7 US students using the classic *Staircase* task that showed only a single configuration of a four-step-high staircase. She found that six of the students drew diagrams of specific staircase heights and then used a recursive approach to add up the columns for each staircase to create the same rule: $n + (n-1) + (n-2) + (n-3) + \cdots + (n-n)$. Clearly, the students recognised the pattern structure but were not able to simplify the rule to a form that permitted the immediate computation of the output value. The remaining two students obtained the functional rule correctly, one by spotting a pattern and the other by using a figural approach. Küchemann (2010), however, argues that the six students' difficulties were attributable to the format of pattern display, and not the type of function. But a recent study by Jurdak and El Mouhayar (2014) seemed to suggest that providing more configurations in a successive manner was not going to make a quadratic generalising task any easier. Over 350 Grades 7 and 8 Lebanese students were asked to make near (Size 5) and far (Size 9 and Size 100) generalisations for both linear and quadratic tasks, each featuring the first four configurations. For linear tasks, although most students were able to reason out the difference between Size 5 and Size 9, they did not add it to Size 5 to obtain Size 9. However, such a level of reasoning was not displayed by most students when the pattern changed from a linear to quadratic relationship, thus suggesting that the students did not find quadratic generalising tasks easy to do.

The student difficulty in dealing with single-configuration generalising tasks described previously is not unique to the US but is a global issue evident in other countries with entirely different cultures and mathematics curricula as well. For instance, Cañadas, Castro and Castro (2011) conducted a study on 359 Grades 9 and 10 Spanish students to explore their construction of far generalisations of both linear and quadratic patterns. They observed a very high success rate of 85% in the task asking for the 234th term of the sequence whose first four terms are 1, 4, 7 and 10. But the success rate dipped to 55% in the task that featured just a single configuration and asked for the number of grey tiles needed to surround a row of 1320 white tiles in the well-known border-tiling task. The numerical approach was prevalently used by the students to make the far generalisations. A similar result for the border-tiling task was also observed in a study by Hoyles and

Küchemann (2001). Nearly 2800 high attaining Year 8 students in the UK were asked to inspect a single generic case in order to find the number of grey tiles needed to surround a row of 60 white tiles. This time, only 42% of the students answered correctly. Taking into account the UK students' prior attainment alongside the fact that a very much smaller value of white tiles was used as compared to the task given to the Spanish students, their success rate was considered low. Although the Spanish students performed better than the UK students, their success rate was somewhat unsurprising given that they were older and perhaps more advanced than their UK counterparts.

Other studies of students' pattern generalisation have investigated young students at the primary level solving for linear patterns. In one of these studies, Stacey (1989) presented three linear generalising tasks to Grades 4 to 8 students in Australia. The tasks, two figural and one numerical, asked the students to find both near and far cases of the patterns. She found that most students were able to identify the patterns and obtain the near cases easily; but they faltered when finding the far cases. The students did not check their generalisations as well to see if they were correct for the particular cases. Stacey noticed the propensity of the students to either count from the drawings they drew and then focus on the numbers in order to make generalisations, or find the common difference between the consecutive terms using tables and then establish the pattern by repeated addition. Warren and Cooper (2008a) worked with 45 Grade 3 Australian students to investigate their ability to describe figural patterns. Most of the tasks they presented showed the first few consecutive configurations whilst a few showed non-successive configurations (for instance, Figures 1, 3 and 5 were given). The students were asked to find the next configuration in the former and the missing configuration (for instance, find Figure 2) in the latter. Over half of them were successful in the former type of questions but there was a widespread failure in the latter type, with almost none of them answering the questions correctly. According to Warren and Cooper, questions that presented the total number of tiles and asked which step this case represented also proved to be testing for the young students.

In a study investigating 40 Grades 5 and 6 Taiwanese students' generalisations in numerical and figural linear generalising tasks, Ma (2009) instructed the students to determine not only the near and far cases of a pattern, but also an expression for its general term. In one of the figural tasks, Ma presented three T-shaped configurations made of dots: five in the first, eight in the second and 11 in the third. She found that over half of the students preferred to convert the figural pattern to a numerical sequence through counting the number of dots in the configurations, then use the resulting numerical sequence to make generalisations. So they ignored the configurations completely instead of analysing them for the pattern structure. This approach led some of these young students to give the general term as $n + 3$, a typical wrong response produced even by older students such as US undergraduates in a study by Rivera and Becker (2005). The remaining students tapped on the figural cues to help them in the generalisations. Not all were, however, successful in establishing the particular cases and the general term. This highlights again that identifying the correct pattern is no guarantee of successful generalisations.

Lin and Yang (2004) examined the generalisations of 1,181 Grade 7 and 1,105 Grade 8 Taiwanese students in a linear and a quadratic figural tasks. In the linear task adapted from the border-tiling task used by Hoyles and Küchemann (2001), 35.4% of Grade 7 and 52.7% of Grade 8 students were successful in determining the number of grey tiles needed to partially surround a row of 40 white tiles when presented with just a single configuration. The students were also asked to determine the number of dots in the 20th configuration of a quadratic pattern that featured the first four configurations made of dots: the first displayed a two-by-two square with a missing dot at a corner, the second a three-by-three square with a missing dot at a corner, and the fourth a five-by-five square with a missing dot at a corner. The success rates for Grade 7 and Grade 8 students were higher at 36.3% and 64.3% respectively.

2.5.2 PERFORMANCE OF SINGAPORE STUDENTS

There has been very little research done on students' performance in pattern generalisation in Singapore despite anecdotal evidence that students often find this topic rather confusing and difficult to learn. Much of what is known about their performance in this topic is drawn heavily from the GCE "O" level and "N" level examiners' reports and the TIMSS reports. The examiners' reports outline students' overall performance for each question, and do not provide any descriptive statistics or detailed information such as the kind of generalising strategies the students used and the different expressions they produced for the general term of a sequence. On the other hand, the TIMSS reports offer descriptive statistics on student responses to the questions. The rest of this section describes some findings in these reports concerning Singapore students' performance in pattern generalising tasks, beginning with those from the examinations, then followed by those from TIMSS.

2.5.2.1 Student performance in examinations

In the "O" level examinations from 1995 to 2009, most of the number pattern questions were numerical generalising tasks, except for two figural generalising tasks. The numerical questions, showing at least the first four terms of a sequence, usually tested students on two skills: *finding a particular term when its position is known* and *deriving an expression for the general term of the sequence*. These questions included both linear and quadratic sequences.

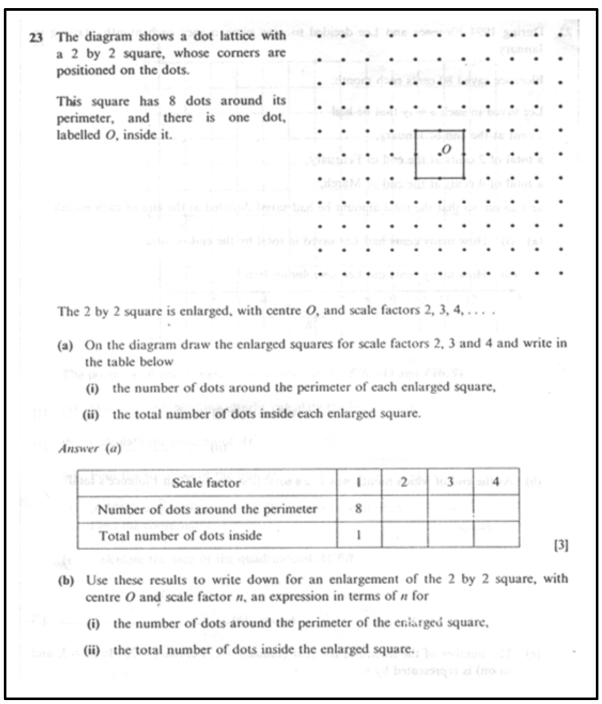
Those questions that dealt with finding a particular term when given its position in the sequence were consistently well answered. The examiners' reports showed that the vast majority of Singapore students knew how to get the next two terms of the quadratic sequence [12, 11, 9, 6, …] in the 1996 examination (University of Cambridge Local Examinations Syndicate, 1997), the 19th term of the quadratic sequence [1, 3, 6, 10, 15, …] in the 2005 examination (Cambridge International Examinations, 2006) and the 12th term of the linear sequence [25, 22, 19, 16, …] in the 2007 examination (Cambridge International Examinations, 2008). Even if the examination question took a different format, many students were similarly successful. Take, for instance, the 2009 examination question: *the first term in a sequence is 38 and each following term is found by subtracting 7 from the previous term.* Although the pattern was not listed as a sequence just like in the previous three questions, the majority of students were still able to find the second and third terms of the sequence correctly. Only a small number of students interpreted the phrase "second and third terms" mistakenly to mean the second and third terms after 38 and so produced 24 and 17 as a result (Cambridge International Examinations, 2010a).

Comparing the questions asking for a specific term when given its position and the general term of the sequence, the latter appears to cause much more of a problem for most "O" level students to deal with because the correct algebraic expression of the general term was rarely established. Taking, for instance, the 2002 examination question involving the linear sequence $[5, 9, 13, 17, 21, \ldots]$, many students gave the incorrect expression $n + 4$ for the n^{th} term (Cambridge International Examinations, 2003a). Similarly, very few students produced the correct expressions for two other linear sequences in subsequent examinations: $[1, 4, 7, 10, 13, \ldots]$ in the 2004 examination and the 2009 examination question described in the last paragraph (Cambridge International Examinations, 2005, 2010a). The wrong answers $n + 3$ and $38 - 7n$ were fairly common to these two questions respectively. Clearly, many students had found the general term of linear sequences far from being straightforward to develop.

If the "O" level students were already grappling with linear sequences, then it is not surprising, perhaps, to find them facing an even greater challenge with the three quadratic generalising questions in the 1996, 2005 and 2007 examinations. All the three questions were indeed poorly done. In the 1996 examination, a significant number of students recognised that they were dealing with the familiar square numbers in the quadratic sequence [4, 9, 16, 25, …], yet could not make the necessary adjustment required. They produced the very popular wrong answer n^2, without realising that the first term was not 1^2, but $(1 + 1)^2$ (University of Cambridge Local Examinations Syndicate, 1997). As for the 2007 question featuring essentially the same quadratic sequence as in the 1996 question except that its first term was 9 instead of 4, the correct functional rule was again rarely seen (Cambridge International Examinations, 2008). Following very much the same outcome as the last two questions, there were also extremely few correct answers to the 2005 question involving the sequence [3, 6, 10, 15, 21, …] (Cambridge International Examinations, 2006).

Interestingly, not all "O" level examination questions requiring students to provide a symbolic expression for the general term were poorly done. The only two figural generalising tasks available over the period between 1995 and 2009 tend to be well answered. Figure 2.11 presents the two figural questions, with (a) from the 1995 examination and (b) from the 1998 examination. The 1995 question is a variation of the popular border-tiling task studied by Taplin and Robertson (1997), and Moss and Beatty (2006).

It can be noticed in Figure 2.11 that the number of dots outlining the rectangles in each question forms a linear sequence. In both questions, the algebraic expressions for the number of dots around the perimeter of any configuration were often correctly derived by the majority of students. This finding supports Rivera and Becker's (2011) comment about the usefulness of figural patterns in facilitating student visualisation of the pattern structure. However, in the last part of the 1995 question which sought the quadratic expression for the number of dots inside the enlarged square, many students performed badly. This indicates that quadratic patterns, regardless of their format of pattern display (that is, whether as a sequence of numeric terms or configurations), prove to be a stern challenge for most students, including even the stronger ones.



23 The diagram shows a dot lattice with a 2 by 2 square, whose corners are positioned on the dots.

This square has 8 dots around its perimeter, and there is one dot, labelled *O*, inside it.

The 2 by 2 square is enlarged, with centre *O*, and scale factors 2, 3, 4, ...

(a) On the diagram draw the enlarged squares for scale factors 2, 3 and 4 and write in the table below

(i) the number of dots around the perimeter of each enlarged square,

(ii) the total number of dots inside each enlarged square.

Answer (*a*)

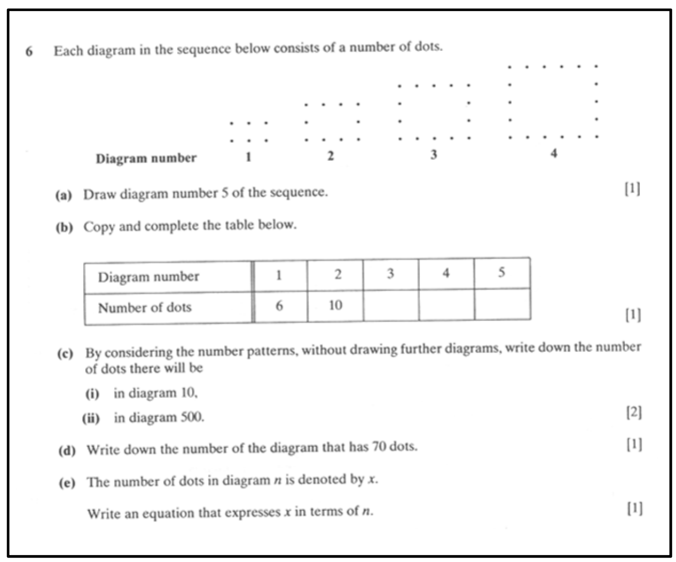
Scale factor	1	2	3	4
Number of dots around the perimeter	8			
Total number of dots inside	1			

[3]

(b) Use these results to write down for an enlargement of the 2 by 2 square, with centre *O* and scale factor *n*, an expression in terms of *n* for

(i) the number of dots around the perimeter of the enlarged square,

(ii) the total number of dots inside the enlarged square.



6 Each diagram in the sequence below consists of a number of dots.

(a) Draw diagram number 5 of the sequence. [1]

(b) Copy and complete the table below.

Diagram number	1	2	3	4	5
Number of dots	6	10			

[1]

(c) By considering the number patterns, without drawing further diagrams, write down the number of dots there will be

(i) in diagram 10,

(ii) in diagram 500. [2]

(d) Write down the number of the diagram that has 70 dots. [1]

(e) The number of dots in diagram *n* is denoted by *x*.

Write an equation that expresses *x* in terms of *n*. [1]

(a) Examination question in 1995 (b) Examination question in 1998

Figure 2.11. Figural generalising tasks in GCE "O" level examination

Turning now to the "N" level examinations, nearly all the number pattern questions over the period from 1997 to 2011 were numerical generalising tasks, and only two were figural generalising tasks. Several of the numerical questions, including the three examples presented below, just simply required students to find a specific term given its position in the sequence.

1. Find the next number in this sequence: $\frac{1}{9}$, $\frac{1}{3}$, 1, 3, 9, … (1997 examination).
2. Write down the next two numbers in the sequence: $13\frac{1}{2}$, 11, $8\frac{1}{2}$, 6, $3\frac{1}{2}$, … (2002 examination).
3. Write down the seventh term in this sequence: 20, $17\frac{1}{2}$, 15, $12\frac{1}{2}$, … (2003 examination).

Questions on deriving the n^{th} term of a sequence were more common in the examinations starting from 2006.

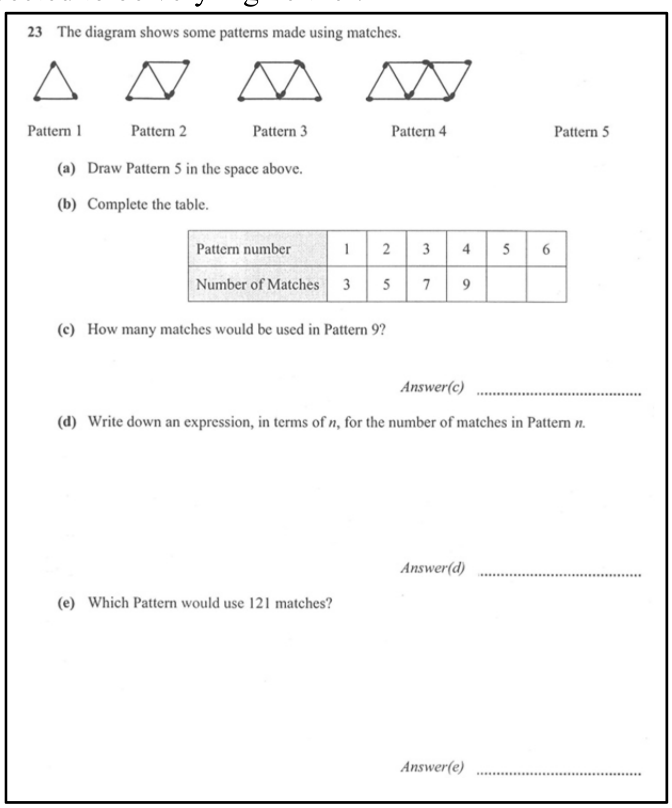
Most "N" level students were able to continue the pattern and give the specific terms in the sequence, but their performance in such questions could occasionally throw up some surprises. Taking the 2002 question for instance, it was thought to be a straightforward question because the next two terms could be determined by taking two successive subtractions of $2\frac{1}{2}$ from $13\frac{1}{2}$. However, so many students recognised what was required to get the first number correct and failed to obtain the correct second number (Cambridge International Examinations, 2003b). Another example is the 2003 question in which some students gave the next seven terms, clearly misunderstanding the meaning of *seventh* (Cambridge International Examinations, 2004).

In very much the same way finding the general term proved to be a difficult question to many "O" level students, such a question also defeated the majority of "N" level students even for the simpler linear numerical sequences. For instance, very few students seemed to have any knowledge of what was required when asked to find an expression for the n^{th} term of the sequence [2, 7, 12, 17, 22, …] in the 2006 examination. As a result, many gave the very common wrong answer $n + 5$ and some even gave other numerical sequences (Cambridge International Examinations, 2007). The problem happened again in the 2011 examination. This time, as the examiners' report pointed out, far more students than expected had difficulty with the general term for the sequence [3, 7, 11, 15, …] (Cambridge International Examinations, 2012). As mentioned earlier, some students used the formula

$a + (n - 1)d$ to find the linear rule but unfortunately made errors in simplifying it (Cambridge International Examinations, 2011).

Based on the GCE "N" level examiners' report, student performance seems to improve when it comes to the only two linear figural questions in the 2009 and 2010 examinations, which are presented in Figure 2.12 below. Many students were able to give the correct algebraic expressions for the n^{th} term although quite a number gave $n + 2$ and $n + 3$ for the 2009 and 2010 questions respectively (Cambridge International Examinations, 2010b, 2011).

Questions asking for the functional rule of a quadratic sequence were, on the other hand, rare in the "N" level examination, with only one in the 2004 examination so far. Unfortunately, the researcher did not manage to obtain the examiners' report for this particular question. Since finding the general term of a linear sequence proves to be a fairly searching question for many students, it might be inferred that finding the general term of a quadratic sequence would be so much more challenging for them. So students' success in this question is not expected to be very high either.



23 The diagram shows some patterns made using matches.

Pattern 1 Pattern 2 Pattern 3 Pattern 4 Pattern 5

(a) Draw Pattern 5 in the space above.

(b) Complete the table.

Pattern number	1	2	3	4	5	6
Number of Matches	3	5	7	9		

(c) How many matches would be used in Pattern 9?

Answer(c) ....

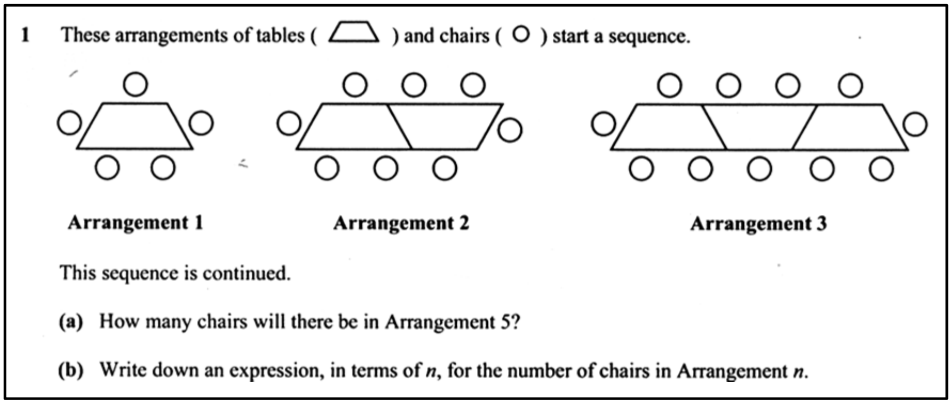
(d) Write down an expression, in terms of n, for the number of matches in Pattern n.

Answer(d) ....

(e) Which Pattern would use 121 matches?

Answer(e) ....

(a) Examination question in 2009


1 These arrangements of tables ( ) and chairs ( ) start a sequence.

Arrangement 1 Arrangement 2 Arrangement 3

This sequence is continued.

(a) How many chairs will there be in Arrangement 5?

(b) Write down an expression, in terms of *n*, for the number of chairs in Arrangement *n*.

(b) Examination question in 2010

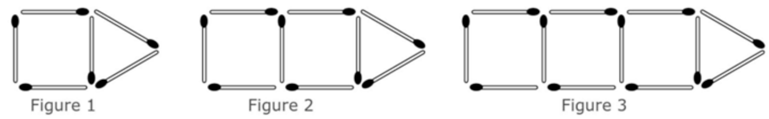
Figure 2.12. Figural generalising tasks in GCE "N" level examination

2.5.2.2 Student performance in TIMSS

This section examines and discusses the TIMSS performance of Year 8 Singapore students in pattern generalising questions. The questions in the 2003, 2007 and 2011 studies covered the same skills as in the GCE "O" level and "N" level examinations. The TIMSS instrument comprised multiple-choice questions, short-answer questions, and structured questions.

The TIMSS–2003 matchstick question in Figure 2.13 below shows three successive configurations and asks Year 8 students to choose from five options the number of matchsticks needed to make Figure 10.

Matchsticks are arranged as shown in the figures.



If the pattern is continued, how many matchsticks would be used to make Figure 10?

Ⓐ 30

Ⓑ 33

Ⓒ 36

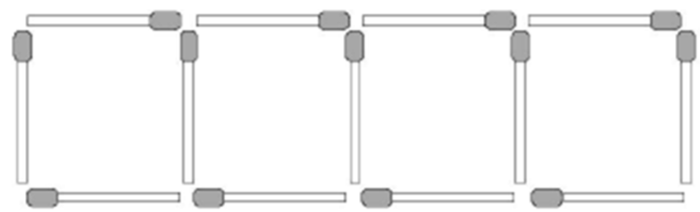
Ⓓ 39

Ⓔ 42

Figure 2.13. TIMSS-2003 matchstick question (ID: M012017)

This question was thought to be a straightforward item because the answer could be verified easily by drawing out the Figure 10 configuration. Yet only 73% of Singapore students chose the correct answer (B) (Martin, 2005). Although they outperformed the Year 8 students internationally (49%), it was still rather unexpected to discover that more than a quarter of the participating Singapore students failed to do it correctly. What is more surprising is that of the four wrong answers, (A) was the most popular response, selected by 11% of the participating students. This answer can be obtained using what Stacey (1989) called the *difference* strategy: that is, take the product of the figure number and the common difference. The students' choice of answer (A) highlights the kind of misconception they have for making far generalisation.

Figure 2.14 presents another matchstick question from TIMSS–2007. A single configuration showing a row of four squares made of 13 matchsticks was provided and students were asked about the number of squares in a row that could be made using 73 matchsticks.



In the figure, 13 matches were used to make 4 squares in a row. What is the number of squares in a row that can be made in this way using 73 matches? Show the calculations that lead to your answer.

Figure 2.14. TIMSS-2007 matchstick question (ID: M032640)

Similar to the border-tiling task investigated by Hoyles and Küchemann (2001), this matchstick question depicted only one configuration and was equivalent to asking students to find the figure number of the given term (i.e., Figure 4 is made using 13 matchsticks. Which figure is made using 73 matchsticks?). As highlighted by Warren and Cooper (2008a), such a task can be fairly tough for many students if they cannot make a link between the number of squares and the number of matchsticks. In view of the result of the border-tiling task, it was then not a surprise to find the TIMSS–2007 question defeating the majority of students: 59% of Singapore students compared to 91% of Year 8 students internationally were unsuccessful (Foy & Olson, 2009). Indeed, the results show that a vast number of Year 8 students did appear to find this TIMSS question extremely tough to cope with. Whether their difficulties are attributable to the single configuration in the question remains unclear. Therefore, it is definitely worth to probe whether providing more configurations in a generalising task will ease student difficulties and improve their student performance.

Figure 2.15 presents the two multiple-choice questions involving rule construction in TIMSS–2003 and TIMSS–2007. In the TIMSS–2003 question, three ordered pairs were provided and students were asked to choose from five options the rule that described how to get the second number in each pair from the first number. The percentages of Singapore students and Year 8 students internationally choosing the correct answer (E) were 71% and

51% respectively. The TIMSS–2007 question, resembling a typical GCE "O" level examination question, provided the first four terms of a sequence and students had to choose the rule that would generate each of these terms. 80% of Singapore students and 63% of Year 8 students internationally picked the correct answer (B). Although the vast majority of Singapore students spotted the patterns that led to the correct rules in both questions, what was disappointing, however, was to find a sizeable number of students failing to identify the correct rule especially when options were provided and could be verified easily. Of the four wrong answers, (D) and (C) were the top choice of Singapore students for the TIMSS–2003 and TIMSS–2007 questions respectively. These findings point to a worrying misunderstanding some students might have. That is, it would suffice to test the truth of a rule using just one or two cases. For instance, the fact that the ordered pair (6, 15) satisfied the rule (D) in the TIMSS–2003 question was enough to convince nearly a fifth of the Singapore students to believe that (D) was the correct answer.

(3, 6) , (6, 15) , (8, 21)

Which of these describes how to get the second number from the first number in every ordered pair above?

(A) Add 3

(B) Subtract 3

(C) Multiply by 2

(D) Multiply by 2 and then add 3

(E) Multiply by 3 and then subtract 3

(a) TIMSS–2003 question (ID: M012029)

2, 5, 11, 23, ...

Starting the pattern at 2, which of these rules would give each of the terms in the number pattern above?

(A) Add 1 to the previous term and then multiply by 2.

(B) Multiply the previous term by 2 and then add 1.

(C) Multiply the previous term by 3 and then subtract 1.

(D) Subtract 1 from the previous term and then multiply by 3.

(b) TIMSS–2007 question (ID: M032273)

Figure 2.15. TIMSS questions involving rule construction

During the writing of this thesis, the TIMSS–2011 results were just released. A similar trend is observed: whilst students are successful in dealing with particular cases of patterns in both numerical and figural pattern generalising tasks, they have considerable difficulty in articulating the functional rule for both linear and quadratic patterns using algebra.

2.6 JUSTIFICATION

In the context of pattern generalisation, expressing generality has been viewed as a key goal of generalising tasks by several researchers (Dreyfus, 1991; Ellis, 2007; Kaput, 1999; Lannin et al., 2006b; Radford, 1996). But the generalisation that is formulated remains only a conjecture until it has been proven to show why it is true (Watson, 1980). Therefore, from a didactic point of view, it is easy to understand why students are often required to justify their generalisations. Furthermore, student justifications can throw light on not only the extent to which the students "see the broad nature of their generalisations [but also] their view of what they deem as a socially accepted justification" (Lannin, 2005, p. 232). So it does also make sense why the generalisation process "cannot avoid the problem of validity" (Radford, 1996, p. 109). As a result, justification is an equally essential and inseparable component of the generalisation process – a view that has been shared by numerous researchers, including Blanton and Kaput (2005), Ellis (2007), and Lannin (2005). To guide the present study, this section reviews relevant research in the area of justification in pattern generalisation to examine what the justification process entails and to identify the various approaches in which students develop their justifications of their rules.

2.6.1 THE NOTION OF JUSTIFICATION

Many different conceptions of justification can be found in the literature. According to Simon and Blume (1996), a mathematical justification is the process of "establishing validity [and] developing an argument that builds from the community's taken-as-shared knowledge" (p. 28). The notion of justification as a means of determining and explaining the truth of a mathematical conjecture or assertion resonates strongly with many other researchers. For instance, Balacheff (1988) described justification as "the basis of the validation of the conjecture" (Balacheff, 1988, p. 225) – a view which garnered Huang's (2005) support as well. To Harel and Sowder (1998, 2007), justification was not just about *ascertaining* the truth of the conjecture or assertion, but also about *persuading* others that it is true, or not. Whilst the process of ascertaining the truth involves removing one's own doubts, the process of persuading is one's attempt to remove others' doubts (Ellis, 2007). However, both processes serve the same purpose, which is to convince people.

The types of justifications expected of students depend on two factors: *the cognitive abilities of students* and *the nature of the task*. For secondary school students, particularly those in the lower grades, a justification does not need to measure up to a geometric proof. Providing a theoretical argument for their conjectures is sometimes not required in the light of their cognitive level until they reach higher level of study (Hoyles & Healy, 1999). In addition, certain mathematical tasks such as pattern generalising tasks do not warrant the use of a theoretical argument. In fact, what students are usually expected to do when asked to justify a conjecture is to explain to others how they arrive at the conjecture and why it is true. Take the Squares task in Figure 2.3(a), for instance, a justification for the rule, $2n + (n + 1)$, could include a description such as "the row of n squares is composed of $(n + 1)$ vertical matchsticks and two horizontal rows of n matchsticks: one on top, and one below. To find the total number of matchsticks, sum up the number of matchsticks in the vertical and horizontal rows". This type of justification, which does not involve any definitions and theorems, is viewed as a less formal argument than a typical mathematical proof (Becker & Rivera, 2009). Nevertheless, it is such a justification, as Lannin (2005) had pointed out, that is valued because it "explains rather than simply convinces" (p. 235), alongside bringing out an essential relation that can be observed across all cases.

2.6.2 JUSTIFICATION SCHEMES

There are two widely used taxonomies in the research literature for examining the justifications of students: one by Simon and Blume (1996), and the other by Harel and Sowder (2007).

The taxonomy of justification schemes by Simon and Blume, drawn from the work of Balacheff (1988), consists of the following four levels in increasing order of sophistication: appeal to external authority, empirical demonstration, generic example, and deductive justification. The *appeal-to-external-authority* justification scheme at the first level is the feeblest way to justify a conjecture, whilst an argument developed from a *deductive* justification scheme at the fourth level is powerful and of utmost value. An *appeal-to-external-authority* justification scheme cites evidence from an authoritative source such as mathematics teachers, textbooks or more knowledgeable peers to substantiate a conjecture. In an *empirical-demonstration* justification scheme at the second level, a conjecture is validated by drawing on previously encountered demonstrations of examples. The third and fourth levels in the taxonomy involve developing justifications based on deduction. A *generic-example* justification scheme expresses a deductive argument in terms of a specific example of a class of cases in question whereas a *deductive* justification scheme at the last level detaches an argument from the use of any particular cases.

Harel and Sowder's taxonomy comprises three broad categories: externally based, empirical and analytic. Externally-based justification schemes reside in some external source, including, for instance, the authority; the form of an argument such as presenting the argument in a two-column format; and the meaningless manipulation of mathematical symbols. Harel and Sowder called the first type of externally-based justification scheme as *authoritarian*, the second type as *ritual* and the last as *symbolic*. Empirical justification schemes, relying solely on examples, are either perceptual or examples-based in nature. A *perceptual* justification scheme uses drawings to validate the truth of a conjecture or assertion. In contrast, an *examples-based* justification scheme validates the conjecture by using one or more specific cases. Finally, analytic justification schemes, regarded by mathematicians and mathematics teachers as offering the ultimate types of justifications, include the transformational and axiomatic approaches to justify a conjecture. A *transformational* justification scheme focuses on the general aspects of a conjecture and involves reasoning that is concerned with validating the conjecture for all cases, rather than on particular instances. This scheme is a necessary precedent to the axiomatic justification scheme. In other words, an *axiomatic* justification scheme is also a transformational one but it further involves the use of accepted mathematical principles and theorems. Of the seven justification schemes described above, five of them appear to be relevant for studying and classifying the justifications of students in pattern generalisation. They are authoritarian, symbolic, perceptual, examples-based and transformational.

The two taxonomies described above share many common features, with the one by Harel and Sowder appearing to discriminate the types of justifications more finely than the other. For instance, the *appeal-to-external-authority* justification scheme in Simon and Blume's taxonomy corresponds to the *authoritarian* justification scheme in Harel and Sowder's taxonomy. However, as Harel and Sowder have pointed out, there are other forms of external sources, apart from the authority such as teachers and textbooks, which can be used to make a justification. Additionally, those two justification schemes are also not valued highly in the respective taxonomies due to their limited explanatory power to convince others of the truth of a conjecture. Another resemblance between the two taxonomies is that Simon and Blume's *empirical-demonstration* justification scheme matches Harel and Sowder's category of empirical justification schemes. Both rely on particular instances to verify the correctness of a conjecture. Lastly, the *deductive* justification scheme in Simon and Blume's taxonomy can be considered a *transformational* one in Harel and Sowder's taxonomy since both arguments are concerned with the general attributes of a conjecture rather than on its specific instances.

Rivera and Becker (2011) had recently introduced four types of justification schemes, based on clinical interviews with students, into Harel and Sowder's category of empirical justification schemes. These four schemes, relevant to the context of pattern generalisation, are *extension generation*, *generic case*, *formula projection* and *formula appearance match*. An *extension-generation* justification utilises more examples to verify the validity of a formula. The *generic case*, similar to Simon and Blume's *generic example*, employs any one particular case to describe the pattern structure. A *formula-projection* justification offers a figural-based argument for a functional rule that involves demonstrating its validity by using the given configurations. On the other hand, a *formula-appearance-match* justification presents a numerical-based argument that involves merely matching and fitting the numerical values corresponding to the figural pattern onto a direct formula.

Rivera and Becker's inclusion of the generic case under the empirical category may lead to disagreement amongst other researchers. Whilst the researchers are not wrong to think of a generic case as offering either a perceptual or examples-based justification, it can however be regarded as a *transformational* one if the underlying structure of a conjecture is perceived and the argument is not only general but involves reasoning as well (Harel & Sowder, 1998). Lastly, their classification scheme is developed on the basis of verbal interviews, which may not cover other justification schemes that students may use to explain their written justifications.

2.6.3 TYPES OF JUSTIFICATIONS STUDENTS USED

Studies investigating the kinds of justifications students offer for pattern generalising tasks reveal that their explanations are often limited to empirical justifications and generic examples (Lannin, 2005; Rivera & Becker, 2011). Lannin (2005) observed that Grade 6 students in his study kept on employing empirical justification to test their functional rules, despite knowing that such a justification was deemed insufficient during classroom discussion. He attributed the popularity of empirical justification to a lack of connection between the symbolic representation of the functional rule and its visual form. He further suggested that students might also be ignorant of the limitation of empirical justification, pointing out that empirical evidence, although it provides assurance about the correctness of a generalisation, lacks the explanatory power needed in an argument to convince others.

Older students appear to use numerical cues differently as they progress to higher level of study, with a shift from using examples to simply verify a generalisation to using them to construct a generalisation. For instance, the justifications of Grade 7 students in Rivera and

Becker's (2011) study demonstrated a widespread use of *formula projection* and *formula appearance match* to explain their generalisations. Generic examples were also spotted in some justifications.

2.7 SUMMARY

Generalisation, highly regarded by many researchers as central to mathematics learning, is essential and useful because of its widespread applications in several mathematical topics as well as its instrumental role in developing algebraic thinking. In the literature, generalisation is more commonly recognised as a process than as a product by many researchers.

The generalisation process, well encapsulated in Ellis' (2007) definition, involves skills ranging from examining the given cases representing a pattern in the generalising task and grasping a commonality amongst the cases in the initial stages to determining particular terms and, finally, expressing a generality for the pattern. However, grasping a commonality is not always obvious to many students. Even if it is, articulating the generality is not inevitably easy too. Indeed, several past studies undertaken in different countries have shown that expressing generality is notoriously elusive for many students. Whilst many of them can often spot the underlying pattern structure in a generalising task, their articulation of an expression of generality is not always guaranteed.

Generalisation takes on several forms, depending on the criterion used. It can be *arithmetic*, *factual*, *contextual* or *symbolic* if the thinking involves objectifying a generality (Radford, 2006). On the other hand, a generalisation examined on the basis of students' use of generalising strategies can be categorised as *numerical* or *figural* (Rivera & Becker, 2008). Two examples of figural generalisation are the *constructive* and *deconstructive* generalisations.

In the corpus of research involving pattern generalisation, there are two common types of generalising tasks aimed at getting students to examine relationships connecting the inputs and outputs and to generate expressions of generality using letters to describe those relationships. First, the *numerical* generalising tasks list a few terms of each pattern sequentially; and second, the *figural* generalising tasks depict each pattern as a sequence of geometric configurations. Extensively investigated in many countries, these generalising tasks not only involve different types of functions, but also appear in a wide range of formats of pattern display.

Most of the studies, undertaken mainly in the west, investigated linear patterns although quadratic patterns involving square or triangle numbers are occasionally tested as well. With limited research on student generalisation of quadratic patterns, the abilities of students in formulating a quadratic rule other than those for the square and triangle numbers, the kind of generalising strategies that they might employ to formulate their rules, and the kind of justifications that they might provide for their rules are all not yet fully clear. Equally uncertain is whether quadratic generalising tasks are more challenging than linear generalising tasks.

Figural generalising tasks used in research studies appear in various formats of pattern display. Most of the tasks depict three or more geometric configurations in a sequential order, but in some cases, the figural patterns are represented using a single configuration, or a sequence of two or three non-successive configurations. What is not yet known, perhaps, is whether generalising tasks with non-successive configurations are harder than those with successive configurations, and whether varying the pattern formats has any effect on students' rule formulation, generalising strategies and justifications.

The literature discusses several generalising strategies for expressing the general rules underlying the patterns presented in generalising tasks. Having emerged from dealing with linear patterns, these strategies range widely from *numerical* types such as *comparison*, *repeated substitution* and the method of difference to *figural* types such as *constructive*, *deconstructive*, *reconstructive* and *figure–ground reversal*. Other types of strategies such as *guess–and–check*, *difference* and *whole-object* are also reported. Results from some studies have confirmed students' use of these strategies to create different generalisations, with certain ones more prevalently employed than the others. For instance, *comparison* and *repeated substitution* strategies are widely used by students, and so is the *constructive* strategy. Students' choice of strategies is often linked to their reasoning as well as their interpretation and discernment of the pattern structure. How do they come to know about the strategy? Why are certain strategies more pervasive than the rest? Which strategies would students believe would best help them to work out the rule? If the relationship underpinning the pattern were to change from a linear to a quadratic rule, would the existing strategies still work or would there be other strategies? Would the best-help strategies that students consider for linear patterns change to suit the latter? These questions would be interesting to explore!

A scrutiny of several studies involving generalising patterns in algebra reveals that two kinds of rules are typically produced by students to describe the pattern structure that they see in a generalising task. One of them is the recursive rule which illustrates the term–to–term change that is used for computing any subsequent term of a pattern when the previous term is known. Whilst a linear recursive rule is relatively easy to produce, the same cannot be said of the quadratic recursive rule. The other type is the functional rule which links a term to its position in the pattern. Its usefulness lies in the direct and quick computation of any term using its position, which is something beyond the facility of a recursive rule when large position number get in the way of students' computation of the term. On the other hand, many studies have shown that formulating a functional rule can be a huge challenge to some students even for a linear pattern, let alone for a quadratic pattern.

For the same generalising task, the use of different generalising strategies can result in structurally different-looking rules. These rules are actually equivalent expressions in different guises with entirely non-equivalent meanings. Scrutinising the equivalent rules that students produce is worthwhile as it serves as a means to gain deeper insight into their thinking, reasoning and discernment of the pattern structure.

The rule defining a pattern can be expressed in many ways and the three modes often used by students are: purely in words, purely in symbols, and in alphanumeric form. A recursive rule is typically written in words. On the other hand, expressing the functional rule symbolically is required normally at the higher level of study after algebra is taught. This is why generalising tasks used in research involving older students ask for a symbolic rule. But when the generalising tasks do not specify the modality of the rule that students are expected to produce, the rule can be articulated in any of the three forms.

Apart from generalisation, another essential component of algebraic activity is justification, which is often used in teaching to elicit thinking and reasoning from students. Viewed as a crucial and inseparable component of the generalisation process (Blanton & Kaput, 2005; Ellis, 2007; Lannin, 2005), it enables teachers to understand how students discern the pattern structure and establish their rules. Rivera and Becker (2011) devised a classification scheme for justification schemes pertaining to pattern generalisation based on verbal justifications given in student interviews. Research findings showed that students often rely on empirical justifications and generic examples when asked to explain the validity of their rules.

Some of the pattern generalisation studies undertaken in the west have discovered that students worked out particular terms for both numerical and figural linear patterns by applying wide-ranging generalising strategies from the numerical and figural approaches to *guess–and–check*. Of these strategies, there is an overwhelming reliance on the numerical approach. Most students were very successful in supplying the next few terms but, predictably, when they were given the harder task of determining the far terms, the success rates tended to dip. Many students were also unable to generalise a linear pattern and formulate the rule. In studies involving generalising tasks that depict the figural pattern with a generic configuration, a significant number of students did appear to face tremendous difficulty with finding particular terms or formulating an algebraic expression for the general term. A prime example is the TIMSS–2007 matchstick task in which a vast majority of Year 8 students internationally were unsuccessful in determining the number of squares in a row that could be made using 73 matchsticks. What is unclear, perhaps, is whether the format of pattern display plays any role in causing the difficulty experienced by students.

Like their international counterparts, Singapore secondary school students fared equally well in finding particular terms of both linear and quadratic patterns but performed poorly when asked to establish a rule for the general term of the patterns. Manifestations of their difficulty in articulating the functional rule occurred normally when the patterns were presented as a sequence of terms. Student success improved when figural patterns were provided, with the functional rule often correctly established by the students.

2.8 RESEARCH FRAMEWORK AND QUESTIONS

As the above literature review has shown clearly, many previous studies on pattern generalisation were undertaken in the west and there were very few reported studies involving East Asian students. One plausible reason for the lack of the latter studies in international English-medium journals is that the East Asian researchers report their studies in journals published in their respective languages: for instance, the researchers in China, Hong Kong and Taiwan publish their work in Chinese-medium journals, those in Japan and Korea in Japanese-medium and Korean-medium journals respectively. Consequently, very little is known about the types of generalising strategies and justification schemes the East Asian students use, as well as the types of rules they formulate for the general term. So the present study seeks to fill in this missing gap about the performance of East Asian students in pattern generalisation by undertaking the research study in Singapore.

Singapore students have consistently done better than their international counterparts in TIMSS generalising tasks but still experience difficulties, particularly at the stage of formulating a symbolic expression for the general term. Recent GCE "O" level and "N" level examiners' reports confirm that the problem in rule construction still remains very much in evidence despite the fact that *number patterns* is not a new topic in the secondary mathematics curriculum and has been taught since the 1990s. Although there are suggestions of Singapore students faring better on figural generalising tasks in the examiners' reports, the current state of their performance in figural tasks is not well understood and studied due to little, if any, local research in this area. Therefore, it is worthwhile to dig into the types of generalising and justification strategies they use, and the types of rules they formulate for the general term alongside the modalities of the different rules. Furthermore, the effect of varying task features such as the format of pattern display and the type of functions on students' rule formulation, generalising strategies and justifications is also not yet clear. The present study is thus framed within such a context to investigate these issues.

Generalisation of patterns is the crux of the present study but with several researchers underscoring the important and complementary role of justification in the generalisation process, it is imperative to examine these two deeply related variables in depth in this present study. An inquiry into the different kinds of generalising strategies adopted by students to formulate their rules could offer insight into their discernment and interpretation of the pattern structure. For this to happen, details about the students' thinking and reasoning in the generalising process must be gathered and one way to collect such information is to engage the students in writing an explanation of how their rules are obtained. Their written justifications are worth analysing to identify the kind of justification schemes used so that a comparison can then be made with those developed by Rivera and Becker (2011) based on verbal justifications. Not only would such an analysis provide insight into students' view of justification, but crucially, it could illuminate why some of the students succeed in deriving the functional rule whilst others do not. Additionally, it would shed light on the link between the students' generalising strategies and justification schemes.

Given the different ways of reasoning and envisioning the pattern structure, the functional rule would take on different equivalent forms. An examination of the various forms of the functional rules would reveal the range of equivalent expressions for the general term that students have created. As mentioned previously in Chapter One, pattern generalisation is introduced after establishing the use of letters as variables to represent unknown numbers in algebraic expressions and formulae. So it would also be worthwhile to look into the modality of the rule to find out how the students express their functional rules and whether they generate a symbolic rule spontaneously.

Two common approaches to representing a figural pattern in a generalising task are found in the research literature. The figural pattern is depicted using three successive configurations or just a single configuration to represent a generic case of the pattern. In addition to the different formats of pattern display, the figural pattern could underpin either a linear or a quadratic function. Thus at the heart of the present study is a systematic investigation of the effect of two task features, namely, the format of pattern display and the type of functions, on students' visualisation of the pattern as well as construction of the functional rule. So task feature is the third variable in the present study, besides generalisation and justification. Findings from this inquiry may then throw light on the influence of the different task features on the students' generalisations and justifications.

The fourth variable to take into account relates to characteristics of the students themselves. The present study will just focus on one student attribute, which is the course the students are enrolled into at the secondary level: that is, the Express and the Normal (Academic) courses. Between the two groups of students, the Express students are academically more able than the Normal (Academic) students. This study decides on investigating the two groups of students at only the Secondary Two level (Year 8, aged 14) because they had learnt *number patterns* in the previous level so their memories might be relatively fresh.

Finally, the corpus of research on pattern generalisation has emphasised greatly on the generalising strategies that students use to establish the functional rule (Lannin, 2005; Lee & Freiman, 2006; Rivera & Becker, 2008; Warren & Cooper, 2008a). But there is very scanty information regarding students' beliefs about the kind of generalising strategies that would best help them to create the rule. Therefore the last variable to be explored in the present study is student belief. By establishing the generalising strategies judged as the most helpful, this study aims to fill in what appears to be a gap in the literature on pattern generalisation. The students' choices of best-help strategies would subsequently be compared with their generalising strategies in the *JuStraGen* test. To provide greater confidence in the outcomes of the investigation on student beliefs, it would be worthwhile to examine whether students would be able to work out the expression for the general term using the best-help strategy of their choice.

To sum up, this study involves the five variables, namely, generalisation of patterns, justification, task features, student characteristic and student belief, and examines the inter-relationships amongst them. All the variables are illustrated in bold boxes in the detailed research framework provided in Figure 2.16, which offers a clear overview of what the entire study is about. As can be seen in the research framework, the inter-relationships between any two variables are indicated by arrows and the attributes to be examined under each variable are represented in boxes attached to the respective variables by dotted lines. Study I investigates two specific issues: first, the Express and Normal (Academic) students' generalisations of figural patterns and justifications of how they make that generalisation, and second, the influence of task features on the students' generalisations and justifications. Study II surveys the kind of generalising strategy that students judge as the most helpful for rule construction and explores the efficacy of their choice of best-help strategy on rule construction.

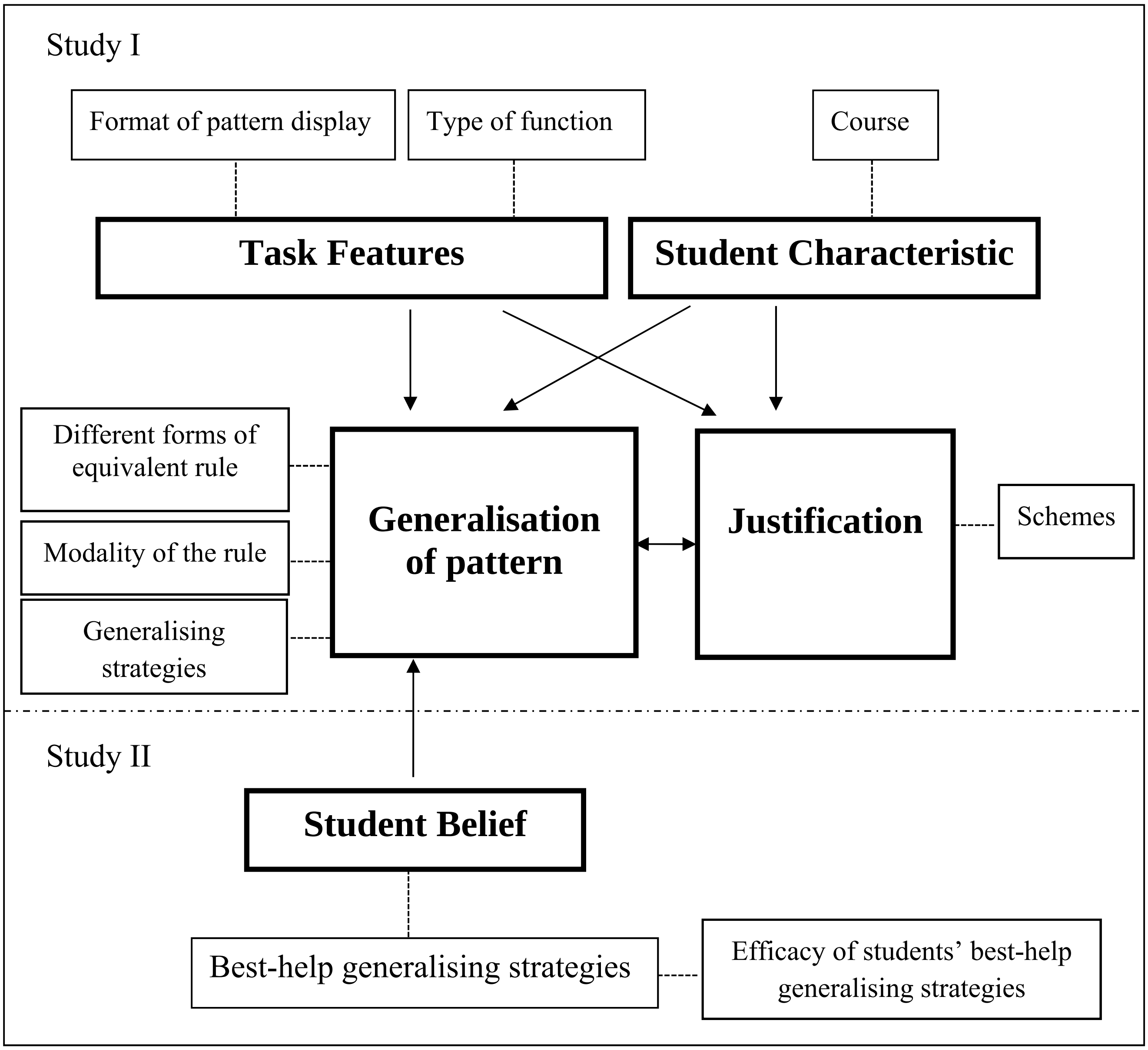


Figure 2.16. Research framework

Based on this research framework presented in Figure 2.16, the specific research questions under each of the four broader research questions are listed as follows:

1. *How do Singapore secondary school students establish the rule that defines a figural pattern?*

   1.1.1 What are the different forms of rules that the students formulate for a figural pattern?
   1.1.2 How do the different equivalent forms of functional rules vary with the different courses the students are enrolled in?
   1.1.3 How do the different equivalent forms of functional rules vary with the different formats of pattern display?
   1.1.4 How do the different equivalent forms of functional rules vary with the different types of function?

   1.2.1 What is the modality of the functional rules that students established?
   1.2.2 How does the modality of the students' functional rules vary with the different courses they are enrolled in?
   1.2.3 How does the modality of the students' functional rules vary with the different formats of pattern display?
   1.2.4 How does the modality of the students' functional rules vary with the different types of function?

   1.3.1 What are the students' generalising strategies for establishing the functional rule?
   1.3.2 How do the students' generalising strategies vary with the different courses they are enrolled in?
   1.3.3 How do the students' generalising strategies vary with the different formats of pattern display?
   1.3.4 How do the students' generalising strategies vary with the different types of function?

2. *How do Singapore secondary school students justify the rule they constructed?*

   2.1.1 What justification schemes do the students adopt to show how they establish the rule?
   2.1.2 How do the students' justification schemes vary with the different courses they are enrolled in?
   2.1.3 How do the students' justification schemes vary with the different formats of pattern display?

2.1.4 How do the students’ justification schemes vary with the different types of function?

3. *How do task features influence Singapore secondary school students’ rule construction?*

3.1.1 Is there any effect of the format of pattern display on the Express students’ rule construction?

3.1.2 Is there any effect of the format of pattern display on the Normal (Academic) students’ rule construction?

3.2.1 Is there any effect of the type of function on the Express students’ rule construction?

3.2.2 Is there any effect of the type of function on the Normal (Academic) students’ rule construction?

4. *What do Singapore secondary school students judge to be the most helpful generalising strategy for constructing the functional rule?*

4.1.1 What generalising strategies do the Express students believe would best help them to generate the linear functional rule?

4.1.2 What generalising strategies do the Express students believe would best help them to generate the quadratic functional rule?

4.2.1 What generalising strategies do the Normal (Academic) students believe would best help them to generate the linear functional rule?

4.2.2 What generalising strategies do the Normal (Academic) students believe would best help them to generate the quadratic functional rule?

4.3.1 Is there any difference in the distribution of students’ choices of best-help generalising strategies between the Express and Normal (Academic) students?

4.3.2 Is there any difference in the distribution of students’ choices of best-help generalising strategies between the successive and non-successive format of pattern display?

4.4 How do the students' choice of best-help generalising strategies compare with their generalising strategies used in the *JuStraGen* test?

4.5 What is the efficacy of the students' choice of best-help generalising strategies on their rule construction?

2.9 CONCLUSION

The insight gained from a comprehensive review of the research literature has helped to identify the following aspects of work for in-depth investigation: (1) the generalising strategies Singapore secondary school students use, (2) the different equivalent rules they construct, (3) the modality of the rules constructed, (4) the schemes they use to justify their generalisations, (5) the effect of two task features on their generalisations and justifications, (6) the generalising strategy that they believe would best help them in rule construction, and (7) the efficacy of their choice of best-help generalising strategy. To shed some light on all these aspects, it is desired to conduct a detailed study of the students' generalisation and justification of patterns. The details of the research design, the participants and their schools, the instruments used in the test, the procedures of administering the test, the scoring rubric and the coding schemes, as well as the plan for data analysis, are reported in the next chapter.

CHAPTER 3 : RESEARCH METHODOLOGY

This chapter begins with revisiting the aims of the present study outlined in Chapter 1. The present study seeks to examine how secondary school students construct and justify the functional rule underpinning a pattern, and explore what they believe to be a helpful generalising strategy for rule construction. However, Chapter 2 has shown that the rule construction process is often fraught with difficulties, with many students often failing to navigate this process successfully. In the light of the literature review presented in Chapter 2, the present study posits that some student difficulties in rule construction are triggered by task features. This study, therefore, seeks to determine the effect of two task features on the students' rule construction. This is the context within which the aims identified in Chapter 1 are framed. The overarching aims of the present study are now restated as follows:

(a) to examine how students make and justify generalisations of figural patterns when the format of pattern display and the type of functions are varied;
(b) to probe systematically the effect of the format of pattern display and the type of functions on students' generalisations; and
(c) to highlight what students believe to be the most helpful generalising strategy.

The present study has two parts: Study I and Study II. The remaining sections of this chapter detail the research design and methodological issues involved in both parts, followed by the profiles of participating students in both the pilot and main studies. It also describes the development of the test instruments, the analytic scoring rubric, the coding schemes and the procedures of data collection. This chapter concludes with a description of the data analysis plan. Figure 3.1 presents an overview of the research methods adopted in this study.

Overall research

	Study I An investigation of students' generalisations and justification of patterns, and the effect of task features on their rule construction	Study II An exploration of what students believe is the the best-help generalising strategy
Aims	To investigate (i) students'generalisations of figural patterns and justifications of how they made the generalisations, (ii) the effect of the format of pattern display and the types of function on students' rule construction.	To explore (i) what students believed to be the most helpful generalising strategy for rule construction, (ii) the efficacy of the students' chosen generalising strategy on their rule construction.
Research design	Experimental research (i) independent-measures design (ii) repeated-measures design	Survey and follow-up student interviews
Subjects	515 secondary school students (aged 14 years, 337 Express and 178 Normal (Academic))	(i) 515 secondary school students for the survey (ii) Of the 515 students, eight Express and eight Normal (Academic) students were selected for the interview
Instruments	(i) a baseline test specially designed to assess students' capabilities in handling algebraic tasks related to pattern generalisation (ii) a test specially designed to assess students' ability to make and justify generalisation	(i) a questionnaire specially designed to probe what students believe is the most helpful generalising strategy (ii) student interviews
Analysis	(i) standard descriptive statistics (ii) standard parametric tests	(i) standard descriptive statistics (ii) standard non-parametric test (iii) qualitative data analysis

Figure 3.1. Summary of the research methods for this study

3.1 STUDY I: AN INVESTIGATION OF STUDENTS' GENERALISATIONS AND JUSTIFICATIONS OF PATTERNS, AND THE EFFECT OF TASK FEATURES ON THEIR RULE CONSTRUCTION

The first part of the present study was to investigate how Singapore secondary school students established a functional rule for predicting any term of a pattern in figural generalising tasks, and justified the way the rule was obtained. At the heart of this study was also a systematic investigation of the effect of task features on students' visualisation of the pattern and, subsequently, construction of the functional rule underpinning the pattern. This section describes the research design for Study I, research instruments, participating students in both the pilot and main studies, and research method.

3.1.1 RESEARCH DESIGN

A common practice to study the effects of certain aspects of an intervention is to undertake an experimental research (Lodico, Spaulding, & Voegtle, 2010). If such a research design is conceived well, it can be a sound and feasible way for achieving the aims of Study I. What has to be done is a careful planning and crafting of the generalising tasks to not only provide insight into students' generalisation and justification, but also permit the inquiry of the effect of task features on students' rule construction.

As mentioned previously in Section 2.8, the two task features under investigation in the present study are (1) the format of pattern display and (2) the type of functions. Both task features co-exist in every generalising task. Take, for instance, the *Squares* task presented in Figure 2.3 above. The pattern is depicted as a sequence of three successive configurations (i.e., the format of pattern display). The rule underpinning this pattern is said to describe a linear relationship (i.e., the type of functions) because the number of matchsticks needed to make a particular configuration is given by the expression, $3n + 2$, where n denotes the corresponding size number. Hence, to measure the effect of task features on students' performance in such tasks, it seems difficult, if not otherwise impossible, to design generalising tasks that focus solely on just one of the two task features because of the interdependence between them. One way to circumvent such a task design issue involving two independence variables is to employ what is known as a factorial design (Burns, 2000). A factorial design is a form of experiment comprising at least two factors, each with a discrete number of levels, and is used to evaluate the effect of interactions between the factors on a dependence variable that is being examined. But due to some implementation constraints, the researcher had to be practical and make realistic decision to modify the factorial design into a mixed design comprising both the independent-measures design as well as the repeated-measures design. In the following two sub-sections, the conception of the research design from the initial to the final design is described in detail.

3.1.1.1 Initial research design

In the initial research design, the dependence variable under examination in the factorial design was the students' success in establishing a functional rule. There were two factors, namely the two task features, and each factor had two levels. For instance, there were two levels of the format of pattern display: successive and non-successive. Similarly, there were also two levels of the type of functions: linear and quadratic. So the factor structure in this 2-factor design is summarised into a table as shown in Table 3.1.

Table 3.1: Factor structure

Factor 1 Format of pattern display	**Factor 2** Type of functions
(a) Successive configurations (b) Non-successive configurations	(a) Linear (b) Quadratic

In this 2-factor design, there were $2 \times 2 = 4$ different treatments. Table 3.2 shows clearly how these four treatments, labelled Treatment 1 to Treatment 4, came about from the pairing of each level of the type of functions with that of the format of pattern display.

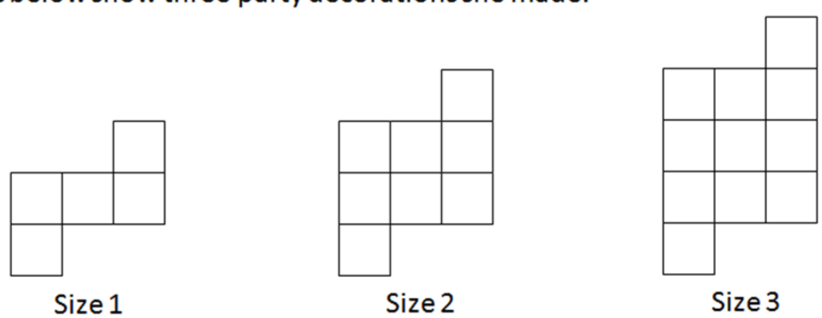
Table 3.2: A 2-factor design

	Format of Pattern Display	
Type of functions	Successive (S)	Non-successive (NS)
Linear (L)	Treatment 1 (L, S)	Treatment 2 (L, NS)
Quadratic (Q)	Treatment 3 (Q, S)	Treatment 4 (Q, NS)

Generalising tasks with task features matching the level conditions stated in the parentheses would be designed for each treatment. A generalising task for Treatment 1 with level condition (L, S) depicts its pattern as a sequence of successive configurations (S) and involves a linear rule (L). The task in Figure 3.2(a), whose rule is $3n + 2$, fits such a level condition perfectly. Figure 3.2(b) shows a generalising task for Treatment 4 whose rule underpinning the pattern presented in a non-successive order (NS) is quadratic (Q): $n^2 + 2(n + 1)$.

Mary used identical square cards to make several birthday party decorations of different sizes.

The diagrams below show three party decorations she made.

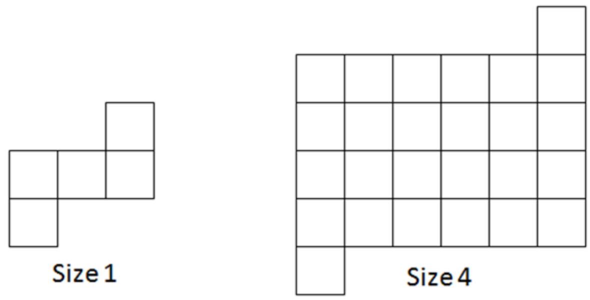


As the size number became larger, more square cards were used.
Mary wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

(a) For Treatment 1

Alice used identical square cards to make several Christmas party decorations of different sizes.

The diagrams below show two party decorations she made.



As the size number became larger, more square cards were used.
Alice wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

(b) For Treatment 4

Figure 3.2. Examples of generalising tasks

A factorial experiment requires a different group of participants to be randomly assigned to each treatment. So in the initial 2-factor design, four different groups of participants would be needed. This criterion of having to assign a different group of participants to each treatment can be tricky and can make the experiment economically infeasible in terms of the available participants (Wiersma & Jurs, 2009). In other words, the experiment cannot proceed when there are relatively few voluntary participants partaking in the study. Of course, one may argue that the experiment could still go ahead with the few available participants or could start only when a targeted number of participants have been reached. Well, these two options can certainly work, but will have some serious drawbacks which make them less than ideal to implement.

The first option of working with a small number of participants might not work out well because the sample size of each treatment will end up becoming even smaller after distributing the participants into four groups. Other than this issue, a small sample would also run a high risk of missing the chance of observing a range of generalising strategies and justification schemes, thus misrepresenting the findings of the present study.

As for the second option, delaying the experiment until a sizeable number of participants are found does not seem to be a wise and practical decision for three reasons. Firstly, the delay may turn out to be a futile wait if eventually the number of participants still falls short of the target number despite all effort to get more participants to partake in the study. Secondly, participants who have initially agreed to partake in the study may lose interest as a result of the delay and then decide to opt out of it. Lastly, participants may not be available anymore for the study due to their tight school schedule and heavy commitments.

In view of all the limitations and constraints concerning this 2-factor design, its feasibility for implementation was reconsidered. After very much deliberation, a decision was reached to modify the factorial design into a new research design so as to make the most effective use of available resources. The next sub-section describes this new research design.

3.1.1.2 Revised research design

The structure of the new research design is presented in Table 3.3 below. As shown clearly, this new research design retained all four treatments in the 2-factor design but reduced the number of groups of participants from the initial four to two (compare with Table 3.2). This marked reduction in the number of groups was especially valuable when there were only a limited number of participants available for the study. This new research design, although very much resembles a factorial experiment, was in fact a mixed design comprising both the independent-measures design and the repeated-measures design. An elaboration of this mixed design follows.

Table 3.3: Revised research design

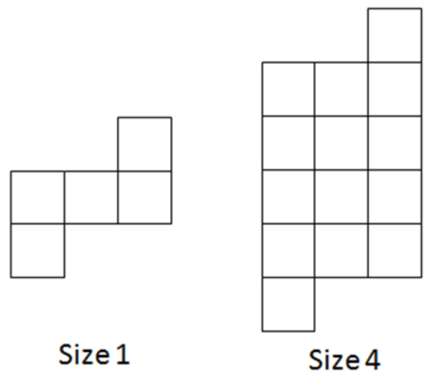
	Format of Pattern Display	
Type of functions	Successive (S)	Non-successive (NS)
Linear (L)	Treatment 1 (L, S)	Treatment 2 (L, NS)
Quadratic (Q)	Treatment 3 (Q, S)	Treatment 4 (Q, NS)
Participants	Group 1	Group 2

An *independent-measures* design involving two separate groups of participants, labelled Group 1 and Group 2, was used to determine whether their success in establishing the functional rule is affected by the format of pattern display. Generalising tasks were designed in such a way that those for the even-numbered treatments were similar to the corresponding ones for odd-numbered treatments except for a change in the format of pattern display. Figure 3.3 below presents a generalising task that not only fits Treatment 2 whose level condition is (L, NS), but also resembles the one for Treatment 1 as shown in Figure 3.2(a). In other words, the participants in Group 1 were given generalising tasks that showed a sequence of successive configurations whereas Group 2 participants received the same tasks, but with configurations in a non-successive order. So to determine whether the format of pattern display had any effect on the participants' success in establishing the functional rule, Treatments 1 and 2 were compared, and so were Treatments 3 and 4.

The participants within each group had to deal with both linear and quadratic generalising tasks involving either successive or non-successive configurations. In other words, the type of functions was the varying factor whilst the format of pattern display remained the constant factor within each group. So a *repeated-measures* design was used to determine whether the participants' success in establishing the functional rule was influenced by the type of functions. Group 1 participants underwent two different treatments, namely Treatments 1 and 3, between which a comparison of performance was conducted to determine whether the type of functions affected their success in constructing the functional rule. In very much the same way, Group 2 participants' performance in Treatment 2 and that in Treatment 4 were also compared to examine the effect of the type of functions on their rule construction.

Mary used identical square cards to make several birthday party decorations of different sizes.

The diagrams below show two party decorations she made.



As the size number became larger, more square cards were used.
Mary wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

Figure 3.3. Generalising task for Treatment 2

3.1.1.3 Strengths of the revised research design

The revised research design has two advantages over the 2-factor design. These are:

(a) ***Number of groups*** The new research design requires fewer groups of participants than the 2-factor design. Instead of having to allocate four different groups of participants for the case of the factorial experiment, this new research design pools the participants into just two groups. In doing so, it makes effective use of human resources. In addition, each participant within the same group undergoes more than one treatment. Thus the participants are employed more effectively. Since the repeated-measures design requires fewer participants (Burns, 2000), the new research design is, therefore, especially useful when relatively few participants are available.

(b) ***Individual differences*** The participants are likely to differ markedly from one individual to another. So individual differences such as personality, attitudes towards mathematics and prior achievements can influence not only the results obtained in a research study, but also the outcome of a hypothesis test. In the case of a factorial experiment involving four different groups of participants assigned to the various treatments, individual differences can be even more evident. This is precisely a crucial disadvantage of the factorial experiment. So this explains why the original research design is abandoned in preference of the revised research design. The use of a repeated-measures design within each group can help to reduce problems caused by individual differences (Burns, 2000).

3.1.2 SUBJECTS

The subjects of this study were 14 year-old students from three co-educational government secondary schools in Singapore. The reason for choosing secondary school students was because they have greater exposure to articulating the functional rule underpinning a pattern that they are examining in words, or in symbolic form than primary school students, as evidenced in the algebra curriculum in the secondary school mathematics syllabus (Ministry of Education (Singapore), 2007). According to the secondary mathematics syllabus, students learn to recognise and represent number patterns, and, more importantly, derive an algebraic expression for predicting any term in the pattern. Therefore, working with secondary school students allows a wider testing scope for exploring their ability to construct the functional rule.

The following two sub-sections are aimed at detailing the profiles of participating students and their respective schools in both the pilot study and the main study for Study I. Before describing these students, there was another small group of 13-year-old students from a secondary school that must be acknowledged here. These Secondary One students, 10 each from the Express and Normal (Academic) courses, participated in a pre-pilot study conducted in May 2009 to gather their feedback about a generalising task, and, primarily, to gauge the amount of time they needed to construct and justify the functional rule of the task. This generalising task was similar to those used in the present study. Their school was not involved in both the pilot study and the main study.

3.1.2.1 Pilot study

Two secondary schools were involved in the pilot study and, for confidentiality, were labelled School P1 and School P2. The main test instrument for Study I was pilot-tested in School P1 whereas the baseline test was trialled in School P2. School P1 is a mixed government school located in the southern part of Singapore. The school caters to the needs of students from Secondary One to Five in all the three courses: Express, Normal (Academic) and Normal (Technical). The PSLE aggregate scores[4] of its students posted to Secondary One Express and Normal (Academic) courses in 2009 ranged from 212 to 242 (median = 217) and 164 to 198[5] (median = 174) respectively. The purpose and the design of the study were explained to the school's Head of Department (HOD) for Mathematics, who is an acquaintance of the researcher.

The main test instrument was supposed to be tested with a group of Secondary One students (aged 13 years) in late October 2009 after their year-end school examinations. However, due to some post-examination school programme, School P1 was unable to arrange for the great majority of the participating students to take the test together during that time. Negotiations with the school then commenced to look for other possible test dates before the school term ended in mid-November 2009, but all efforts were unsuccessful. As a result, the pilot test had to be postponed to March 2010. By then, the students had already progressed to Secondary Two. In the remaining section, more details about the participating students are revealed.

The subjects in the pilot study were 14-year-old Secondary Two students from the Express and Normal (Academic) courses in 2010. These students were selected by the HOD for Mathematics according to their Mathematics grades at PSLE. The Express students were selected from four classes whilst the Normal (Academic) students came from two classes.

The main test instrument for Study I was trialled with 45 students on two separate days, which were two days apart. There were 29 Express students and 16 Normal (Academic) students. 23 of the students were girls and the remaining 22 were boys. The students were divided as evenly as possible into two different groups, Group 1 and Group 2, based on their PSLE Mathematics grades and gender. Table 3.4 below shows the distribution of Group 1 and Group 2 students by course, gender and PSLE Mathematics grades. The distributions of Express and Normal (Academic) students in the two groups by PSLE

4 The PSLE aggregate score refers to the total score obtained for English Language, Mother Tongue, Mathematics and Science at the Primary School Leaving Examination.

5 The highest PSLE aggregate score each year is about 285. To qualify for the Express course, a student has to obtain an aggregate score of at least 200. Students scoring between 150 to 199 are placed in the Normal (Academic) course. Those scoring below 150 are placed in the Normal (Technical) course.

Mathematics grades and by gender were similar. Group 1 had 23 students (11 girls and 12 boys). 8 of them scored A* or A for Mathematics at PSLE and 15 obtained B or C. Group 2 had 22 students (12 girls and 10 boys). 7 of them obtained A* or A for Mathematics at PSLE and 15 scored B or C.

On both test days, Group 1 students were given generalising tasks that show a sequence of successive configurations whereas Group 2 students received the same tasks, but with configurations presented in a non-successive order. At the time of the pilot study, these students had already learned how to recognise patterns and work out the functional rule underpinning each pattern. The main test instrument will be described below in Section 3.1.3.2.

Table 3.4: Distribution of students in School P1 by course, gender and PSLE Mathematics grades

		PSLE Mathematics grades					Gender	
		A*	A	B	C	Total	Girls	Boys
Express	Group 1	2	6	4	2	14	7	7
	Group 2	2	5	5	3	15	8	7
Normal (Academic)	Group 1	0	0	5	4	9	4	5
	Group 2	0	0	4	3	7	4	3

With more students involved in the main study, it became evident that a baseline test was necessary to add an additional measure to solely using the PSLE Mathematics grades to minimise individual differences between two groups of students. This viewpoint emerged following the PhD upgrading examination in November 2010 during which the examiners commented that with a time lapse of more than a year between PSLE and the implementation of the test instruments, the use of the PSLE Mathematics grades alone was insufficient and might raise concern about group homogeneity. Hence a baseline test was specially developed between December 2010 and January 2011 to offer confidence in the results of the present study. This baseline test will be described later in Section 3.1.3.1.

Trialling the baseline test were 23 Secondary Two students (14 years) from School P2, a mixed government school located in the central part of Singapore. School P2 offers Express and Normal (Academic) courses from Secondary One to Five. The PSLE aggregate scores of this cohort of Secondary Two students ranged from 227 to 244 (median = 229) for the Express course and 183 to 199 (median = 186) for the Normal (Academic) course. The students, 15 from the Express course and eight from the Normal (Academic) course, volunteered to participate in the testing of the baseline test in April 2011.

3.1.2.2 Main study

Twelve secondary schools were shortlisted from all the secondary schools in Singapore on the basis of the PSLE aggregate scores of their Secondary One cohorts in the year 2010. Schools with aggregate scores close to those of School P were selected. There were two reasons why the PSLE aggregate scores of the 2010 Secondary One cohorts were considered in the school selection process. First, PSLE is a national examination, so the aggregate scores become a reasonable measure for selecting schools with comparable cohorts of students. Second, the Secondary One students in 2010 would become the subjects of the main study in 2011. These schools were located in different parts of Singapore, with three in the north, one in the east, two in the south and six in the west.

The twelve schools were first approached through email between January and February 2011. The purpose and the design of the study were explained to the principals or the HODs for Mathematics of these schools. Personal visits to meet the HODs of five schools who showed interest then followed. During the visit, a presentation giving detailed information about the study was done, further questions about the study were clarified and administrative issues were also discussed. Eventually, only three schools agreed to participate in this study. A formal letter giving detailed information about the study was then sent to the principals and the HODs of the three schools. For confidentiality, these three schools are labelled School M1, School M2 and School M3. All three are mixed government schools. School M1 is located in the north whilst Schools M2 and M3 in the west. Table 3.5 below presents the PSLE aggregate scores for the Secondary One cohorts in these three schools and the pilot school.

Table 3.5: PSLE Aggregate Scores of Secondary One cohorts in main and pilot studies

	2010 Secondary One					
	Express			Normal (A)		
	L	U	M	L	U	M
School M1	216	256	222	171	199	180
School M2	220	239	224	182	199	187
School M3	229	245	234	181	199	187
	2009 Secondary One					
School P1	212	242	217	164	198	174
	2010 Secondary One					
School P2	227	244	229	183	199	186

L: Lower PSLE aggregate score; U: Upper PSLE aggregate score; M: median PSLE aggregate score

The table shows that the students in both the pilot study and the main study were fairly similar in terms of PSLE aggregate scores although the Express students from School M3 did appear to be academically more able than their counterparts from Schools M1 and M2 because of the higher PSLE aggregate score.

The subjects in the main study came from the Express and Normal (Academic) courses in the three schools, Schools M1, M2 and M3. They were 14-year-old Secondary Two students when they participated in the study in 2011. These students were selected by the HODs for Mathematics of the respective schools. Schools M1 and M2 picked three intact classes from each course whilst School M3 chose three intact Express classes and one intact Normal (Academic) class for the study. In all three schools, written consent was obtained from the HODs acting *in loco parentis*. The students were also informed by their schools at the outset of the study that they had the right to withdraw at any time.

At the start of Study I, there were 612 students but two constraints led to a reduction of the sample size. 97 students were absent for at least one test or did not complete at least one test or one questionnaire. Consequently, the final sample of Study I was 515 students. Table 3.6 shows the sample sizes at the start and end of Study I. As the table reveals, there was a small loss of 25 students (4.1%) in the Express course whereas the reduction in sample size was considerable in the Normal (Academic) course, with a loss of 72 students (11.8%). There are a number of possible reasons why the latter had happened. First, getting the Normal (Academic) students to stay back for any activities after school was never an easy task. The present study was no exception. Whilst the majority of students were very cooperative, some failed to attend the test despite their teachers' reminders. Second, some of the students did not treat the test seriously because they knew that the tests were not part of their school assessment. Finally, some students might have regarded those two days that they had to stay back after school for about an hour each as a waste of time. As a result, some skipped at least one test and were, therefore, marked as absent. Others chose not to answer the generalising tasks or the questionnaire even though they were present for the test.

Table 3.6: Sample sizes before and after Study I

	Start of Study I			End of Study I		
	Express	Normal (Academic)	Total	Express	Normal (Academic)	Total
M1	120	126	246	104	94	198
M2	120	85	205	112	53	165
M3	122	39	161	121	31	152
Total	362	250	612	337	178	515

Table 3.7 shows the distribution of the sample by course, format of pattern display, gender and school. As the table shows, two groups of students (Group 1 and Group 2) were formed. Group 1 students were given generalising tasks that showed a sequence of successive (S) configurations whereas Group 2 students received the same tasks, but with configurations in a non-successive (NS) order. Each group included students from each course and school.

There were 266 students in Group 1 and another 249 in Group 2, giving a total of 515 students in the Study I. Of the 515 students, 337 of them were in the Express course and the remaining 178 in the Normal (Academic) course. The percentages of Express and Normal (Academic) students were 65.4% and 34.6% respectively. These values were rather close to those in the Secondary One population when compared with the 2010 statistics (Express: 70.6%, Normal (Academic): 29.4%) (Ministry of Education (Singapore), 2011).

There were 273 girls and 242 boys taking part in this study, of which 143 girls and 123 boys were in Group 1, and 130 girls and 119 boys in Group 2. The percentages of girls (53.0%) and boys (47.0%) in overall were also close to those in the Secondary One population in the Express and Normal (Academic) courses when compared with the 2010 statistics (Girls: 50.4%, Boys: 49.6%) (Ministry of Education (Singapore), 2011).

Table 3.7: Distribution of the subjects by course, format of pattern display, gender and school

		Express				Normal (Academic)				Total			
		M1	M2	M3	Total	M1	M2	M3	Total	M1	M2	M3	Total
S	Girls	30	34	29	93	25	18	7	50	55	52	36	143
(Group 1)	Boys	24	22	31	77	27	10	9	46	51	32	40	123
	Total	54	56	60	170	52	28	16	96	106	84	76	266
NS	Girls	25	36	28	89	20	15	6	41	45	51	34	130
(Group 2)	Boys	25	20	33	78	22	10	9	41	47	30	42	119
	Total	50	56	61	167	42	25	15	82	92	81	76	249
Total			337 (65.4%)				178 (34.6%)				515 (100%)		
2010 Statistics			29 785 (70.6%)				12 394 (29.4%)				42 179 (100%)		
	Girls		15417				5832					21 249 (50.4%)	
	Boys		14368				6562					20 930 (49.6%)	

A description of how the participating students in the three schools were divided into Group 1 and Group 2 will now follow. Students were allocated to the different groups on the basis of the following criteria:

(a) their gender;

(b) their scores in the baseline test, called *Generalisation Attainment Test* (*GAT*), which is detailed in Section 3.1.3.1;

(c) their PSLE Mathematics grades.

To get an even distribution of the girls and the boys in Group 1 and Group 2, the students in each course were first sorted by gender into two groups. Within each group, the students were ranked, in descending order, first according to their *GAT* scores, then followed by their PSLE Mathematics grades. After sorting in this manner, the topmost student was one with the highest *GAT* score and the best PSLE Mathematics grade. Students in the odd-numbered rank positions were assigned to Group 1 and the rest to Group 2. To check the compatibility of these two groups of students for each course, the distribution of students by the format of pattern display, PSLE Mathematics grades and school was subsequently examined and the mean *GAT* scores computed.

Table 3.8 shows the distribution of Express students by the format of pattern display, PSLE Mathematics grades, school, as well as the mean *GAT* scores and standard deviation. There were 170 students in Group 1 and another 167 in Group 2, giving a total of 337 students. Three of these students did not have any PSLE Mathematics grades because they were foreign students who joined their schools directly without sitting PSLE. So they were classified under *Others*. The distributions of Group 1 and Group 2 students in overall by PSLE Mathematics grades were similar. Further, the mean *GAT* scores of 43.4 (sd = 5.5) for Group 1 overall and 43.5 (sd = 5.2) for Group 2 overall were also close. With comparable distribution of PSLE mathematics grades and mean *GAT* scores between Group 1 and Group 2 students in overall, these two groups were assumed to be as equivalent in terms of academic abilities as the researcher could make them.

Table 3.8: Distribution of Express subjects by format of pattern display, PSLE mathematics grades, school, mean GAT scores and standard deviation

			PSLE Mathematics Grade					*GAT*	
Schools	Pattern format	N	A*	A	B	C	Others	Mean	sd
M1	S	54	5	31	14	3	1	43.0	5.1
	NS	50	4	28	15	2	1	42.6	4.9
M2	S	56	6	29	21	0	0	42.0	5.9
	NS	56	6	28	22	0	0	41.7	5.7
M3	S	60	15	34	9	2	0	45.2	5.1
	NS	61	16	34	10	0	1	46.0	3.9
Total	S	170	26	94	44	5	1	43.4	5.5
	NS	167	26	90	47	2	2	43.5	5.2

Table 3.9 shows the distribution of Normal (Academic) students by the format of pattern display, PSLE Mathematics grades, school, as well as the mean *GAT* scores and standard deviation. Group 1 and Group 2 in overall had 96 and 82 students respectively, giving a total of 178 students. Like in the Express course, two foreign students were classified under *Others* as they joined their schools directly without sitting PSLE. Despite a rather wide difference of 11 students in Grade C, the distributions of the two groups of students in overall by PSLE Mathematics grades were, nevertheless, still fairly similar. Additionally, the mean *GAT* scores of 30.4 (sd = 7.6) for Group 1 overall and 29.9 (sd = 8.1) for Group 2 overall were close as well. Hence it is reasonable to assume that these two groups were also comparable in terms of academic abilities.

Table 3.9: Distribution of Normal (Academic) subjects by format of pattern display, PSLE mathematics grades, school, mean GAT scores and standard deviation

Schools	Pattern format	n	PSLE Mathematics Grade						*GAT*	
			A	B	C	D	E	Others	Mean	Sd
M1	S	52	1	13	22	11	5	0	27.8	7.5
	NS	42	2	13	13	9	5	0	28.0	8.3
M2	S	28	0	24	1	2	1	0	34.0	6.9
	NS	25	1	19	0	2	1	2	31.3	8.2
M3	S	16	1	12	3	0	0	0	32.3	6.2
	NS	15	1	12	2	0	0	0	33.1	5.9
Total	S	96	2	49	26	13	6	0	30.4	7.6
	NS	82	4	44	15	11	6	2	29.9	8.1

Tables 3.8 and 3.9 show that the Express and Normal (Academic) student samples in the main study were spread over a range of learning abilities with respect to the PSLE Mathematics grades. The results from such a spread of learning abilities should, therefore, give a good notion of learning difficulties, the generalising strategies, and the justification schemes of the student population in both the Express and Normal (Academic) courses in general.

At the time of the present study, the participating students should have learnt the topic of number patterns in the Singapore mathematics curriculum when they were in Secondary One. These students should be able to continue for a few more terms any pattern presented as a sequence of numbers or figures, make a near and far generalisation and establish, in the form of an algebraic expression, the functional rule for predicting any term. Further, they should also be far more familiar in dealing with linear patterns than with non-linear ones, which are less commonly featured in their mathematics textbook.

3.1.3 INSTRUMENTS

The data for Study I was collected using one research instrument: *Strategies and Justifications in Mathematical Generalisation* (*JuStraGen*) Test. Before administering this instrument, the participating students took a baseline test called the *Generalisation Attainment Test* (*GAT*). As the literature review in Chapter 2 has shown, there were no test instruments that entirely inquired into the effect of the format of pattern display and the type of functions on students' rule construction, and students' competence in pattern-related algebraic tasks. Given this situation, the *JuStraGen* and *GAT* test instruments had to be developed specifically to achieve the overarching aims of the present study. In the next few sections, the details of these two totally new test instruments will be described completely.

3.1.3.1 Generalisation Attainment Test (*GAT*)

The *GAT* assesses students' capabilities in handling algebraic tasks related to the generalisation of number patterns. It is a 50-mark paper-and-pencil test consisting of 10 multiple-choice questions, 13 short-answer questions and two structured questions designed to gauge the competence level of the participating students in dealing with certain algebraic questions. The test duration was one hour and calculators were allowed. Students were required to answer all questions. *GAT* was conducted before administering the research instrument for Study I: the *JuStraGen* test. The *GAT* instrument and its answer keys appear under Appendices 2(a) and 2(b).

As previously explained in Section 3.1.2.1, the development of *GAT* was motivated by the need for a baseline test to detect whether there was any individual differences between different groups of subjects. For any experimental research design involving different groups of subjects, this condition is particularly essential to make sure that the results of the research study can be interpreted confidently.

In devising the *GAT* instrument, an attempt had been made to cover a wide range of typical test items found in the mathematics textbooks used in Singapore secondary schools. The choice of item content was identified from a range of sources such as the literature review in Chapter 2, the Singapore mathematics curriculum and textbooks, as well as the researcher's experience as a secondary school teacher. The test questions in the *GAT* instrument included fundamental algebraic skills such as

(a) translate an expression in words into an algebraic expression;
(b) manipulate algebraic expressions;
(c) evaluate an unknown output value when given the formula and an input value;
(d) evaluate an unknown input value when given the formula and an output value;
(e) generate terms of simple number sequences when given the rule;
(f) generate terms of simple number sequences when given the initial terms;
(g) state the rule of a pattern.

During the development of the *GAT* instrument, it soon became clear that a great many test questions had to be devised to encompass all the above-mentioned skills thoroughly. Additionally, the time interval between the *GAT* test and the *JuStraGen* test was only slightly more than a week. Within this short period of time, the marking of all *GAT* scripts had to be completed, the *GAT* scores entered into a spreadsheet, the students sorted into two groups, and, finally, the participating schools informed of the student groupings so that the administrative logistics for conducting the *JuStraGen* test could be taken care of. A format for the *GAT* instrument, therefore, had to be constructed to take into account the number of test questions needed and the time constraints of marking. It was decided to use a mixture of multiple-choice questions alongside the short-answer questions and structured questions, which are the two kinds of question types commonly used in secondary school mathematics tests and examinations.

The *GAT* instrument used three question types: (1) multiple-choice questions, (2) short-answer questions, and (3) structured questions. The first two types of questions usually carry only a few marks per question, thus many questions can be set. In the *GAT* instrument, there were 10 multiple-choice questions each worth two marks and 13 short-answer questions each worth varying marks that range from one to three marks. For pragmatic reasons such as the constraints on students to answer several questions within a stipulated time and on the researcher to complete the marking of all scripts quickly, it would be very unwise to design too many structured questions. So there were only two such questions in *GAT*, each carrying four marks.

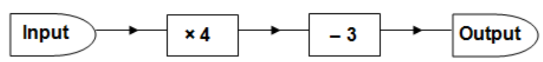
All test items were kept as short and simple as possible. Apart from this, the numerical quantities given in the test items were also kept as small and manageable as possible for the students to work on even without the aid of any calculator.

For each multiple-choice question, five options were given and only one of them was the correct answer. The task of designing such questions was a difficult, yet rewarding, one because considerable effort had gone into making the four distractors as plausible and attractive as possible to the students. Ideas for these distractors were drawn from the researcher's knowledge of the common student misconceptions and mistakes. Table 3.10 below presents an example of a multiple-choice question with accompanying explanation of how the distractors were being created.

Table 3.10: Multiple-choice question with explanation for distractors

3. Given that $S = 4r^2$, the value of S when $r = 3$ is	
(A) 13	By misinterpreting $4r^2$ as $4 + r^2$ instead of $4 \times r^2$, the value of S is $4 + 3^2$ = 13.
(B) 24	By misinterpreting 3^2 as 6 instead of 9, the value of S is 4 × 6 = 24.
(C) 36	Substituting $r = 3$ into $S = 4r^2$, the value of S is 4×3^2 = 36. The correct answer.
(D) 49	By misinterpreting $4r^2$ as $(4 + r)^2$ instead of $4 \times r^2$, the value of S is $(4 + 3)^2$ = 49.
(E) 144	By misinterpreting $4r^2$ as $(4r)^2$ instead of $4(r^2)$, the value of S is $(4 \times 3)^2$ = 144.

Table 3.11: Short-answer and structured questions

<table>
<tr><th>Question Type</th><th>Skill tested</th><th>Question</th></tr>
<tr><td rowspan="5">Short-answer question</td><td>translate an expression in words into an algebraic expression</td><td>11. Express the statement below as an algebraic expression.
Subtract 7 from twice of x.</td></tr>
<tr><td>manipulate algebraic expressions</td><td>16. Simplify $5n - 9 - 3n + 4$.</td></tr>
<tr><td>evaluate an unknown output value when given the formula and an input value</td><td>14. Given the formula $T = \frac{1}{2}n(n+1)$, find the value of T when $n = 20$.</td></tr>
<tr><td>generate terms of simple number sequences when given the initial terms</td><td>20. Fill in the **two** missing terms in the following sequence:
23, _____, 39, 47, _____, 63, …</td></tr>
<tr><td>generate terms of simple number sequences when given the rule</td><td>22. The n^{th} term of a number sequence is $7 - 3n$.
Write down the **first three** terms of this sequence.</td></tr>
<tr><td rowspan="2">Structured question</td><td>state the rule of a pattern</td><td>25. John uses the following rule to obtain the values in the table below:
<table><tr><th>Input</th><th>Output</th></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>5</td></tr><tr><td>3</td><td>9</td></tr><tr><td>4</td><td>13</td></tr></table>(a) What is the output when the input is n?</td></tr>
<tr><td>evaluate an unknown input value when given the formula and an output value</td><td>(b) Find the input value that John will have to use so as to obtain the output value of 77.</td></tr>
</table>

Each short-answer question tested students on only one of the skills listed above. Students needed to complete a blank or supply either a number or an algebraic expression that answered a question. On the other hand, each of the two structured questions set consisted of two parts related to the same stem. But the two parts were independent of one another and need not be worked out in the order that they were presented. Examples of these two types of questions are provided in Table 3.11 above.

A prototype version of the *GAT* instrument together with its answer keys and table of specification were passed to one expert in mathematics education at the National Institute of Education and a retired HOD for mathematics for examining the clarity and validity of the test questions. These two persons were very experienced and familiar with the kinds of questions covered in the algebra syllabus at the secondary level. They were satisfied that the test covered all the algebraic skills to be tested and the test questions were appropriately written in general for Secondary Two students. They did, however, suggest two improvements for four test questions. First, Questions 1 and 7 should be reworded to omit the term *equivalent* because the main objective of these two questions was to test students on the simplification of algebraic expressions, and *not* the notion of equivalence. Table 3.12 shows the original and revised versions for Questions 1 and 7.

Table 3.12: Modifications to Questions 1 and 7

Original question	Revised question
1. Which *one* of the following terms is equivalent to $mn - 2mn + 3mn + 6mn$? **(A)** 8 **(B)** $8mn$ **(C)** $9mn - 2$ **(D)** $9m^2n^2 - 1$ **(E)** $10mn$	1. $mn - 2mn + 3mn + 6mn$, when simplified, is equal to **(A)** 8 **(B)** $8mn$ **(C)** $9mn - 2$ **(D)** $9m^2n^2 - 1$ **(E)** $10mn$
7. Which *one* of the following algebraic expressions is **NOT** equivalent to the others? **(A)** $1 + 3(n - 1)$ **(B)** $2(2n - 1) - n$ **(C)** $2(n - 2) + (n - 1)$ **(D)** $(2n - 1) + (n - 1)$ **(E)** $n(2n - 1) - 2(n - 1)^2$	7. Which *one* of the following algebraic expressions does **NOT** simplify to the same answer? **(A)** $1 + 3(n - 1)$ **(B)** $2(2n - 1) - n$ **(C)** $2(n - 2) + (n - 1)$ **(D)** $(2n - 1) + (n - 1)$ **(E)** $n(2n - 1) - 2(n - 1)^2$

Second, the blanks in Questions 20 and 23 should be shortened. One of the experts felt that the length of the original blanks might mislead students to fill in more than one missing terms per blank. Other than these two suggestions, no further revision was required.

What follows in the remaining section is a discussion of the students' performance in *GAT*. In Section 3.1.2.2, Table 3.8 shows that the Group 1 Express students from School M3 had the highest mean *GAT* score and their counterparts from School M2 had the lowest. The mean *GAT* score for the Group 2 Express students reflected a similar trend. Like their mean PSLE aggregate score, the students from School M3 also topped the mean *GAT* score. For students from Schools M2 and M3, one would expect their mean *GAT* scores to mirror their mean PSLE aggregate scores as well. Yet this was not the case. Students from School M2 had a lower mean *GAT* score than their counterparts from School M1 despite having a higher mean PSLE aggregate score. Since the mean PSLE aggregate scores of students from Schools M1 and M2 differed by only two points and, furthermore, the difference in their mean *GAT* scores was also very slight, the variation in the mean *GAT* scores was believed to be reasonable. Therefore the trend in the mean *GAT* scores of the Express students from all three schools was fairly consistent with their mean PSLE aggregate scores (see Table 3.5). The mean *GAT* score for Group 1 Express overall was 43.4 (sd = 5.5) whilst that for Group 2 Express overall was 43.5 (sd = 5.2). The two means were almost equal, suggesting that the two groups of Express students were comparable in terms of academic abilities.

The mean *GAT* scores of the Normal (Academic) students followed a somewhat similar situation as their Express counterparts. Table 3.9 in Section 3.1.2.2 shows that the Group 1 Normal (Academic) students from School M2 had the highest mean *GAT* score and their counterparts from School M1 had the lowest. On the other hand, the Group 2 Normal (Academic) students from School M3 had the highest mean *GAT* score and their counterparts from School M1 again had the lowest. In other words, students from Schools M2 and M3 outperformed their counterparts from School M1 in general. This outcome is hardly surprising because it reflects the same trend observed in the mean PSLE aggregate scores of the Normal (Academic) students from all three schools (see Table 3.5). In short, the mean *GAT* scores of the Normal (Academic) students from all three schools were fairly consistent with their mean PSLE aggregate scores. The mean *GAT* scores for Group 1 Normal (Academic) overall and Group 2 Normal (Academic) overall were 30.4 (sd = 7.6) and 29.9 (sd = 8.1) respectively. Like in the Express course, the two means were very close, thus suggesting that the two groups of Normal (Academic) students were comparable in terms of academic abilities.

3.1.3.2 Strategies and Justifications in Mathematical Generalisation (*JuStraGen*) test

As the literature review in the previous chapter had shown, there was presently no test instrument that entirely inquired into the effect of task features on students' pattern recognition and their ability to generalise. Given this situation, a totally new test instrument has to be developed specifically to achieve the aims of the present study. So this was how the instrument, entitled *Strategies and Justifications in Mathematical Generalisation* (*JuStraGen*), was developed.

The *JuStraGen* test provides an assessment of students' ability to generalise figural pattern tasks, as well as a measurement of the effect of two task features on their rule construction. It is a paper-and-pencil test that consists of eight generalising tasks designed to investigate how students construct and justify the functional rule for predicting any term of a pattern in the tasks. Of the eight tasks, four involved an underlying linear pattern structure whilst the rest a quadratic structure. In addition, the test was also developed specially to examine systematically the effect of *the format of pattern display* and *the type of functions* on students' ability to construct the functional rule. Each task existed in two different formats, with its pattern depicted as (1) a sequence of three successive diagrams, and (2) a single diagram or a sequence of two or three non-successive diagrams.

The design of the *JuStraGen* test was guided by the revised research design previously discussed in Section 3.1.1.2 above and the following general considerations.

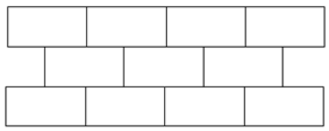
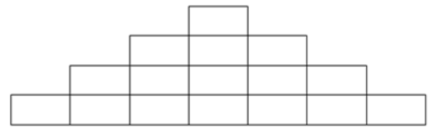
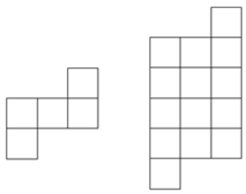
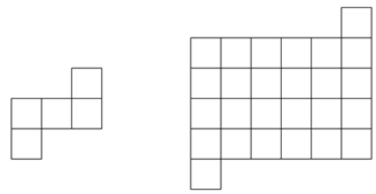
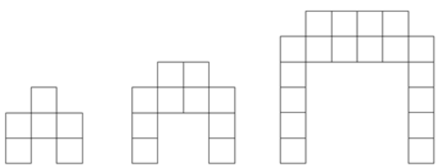
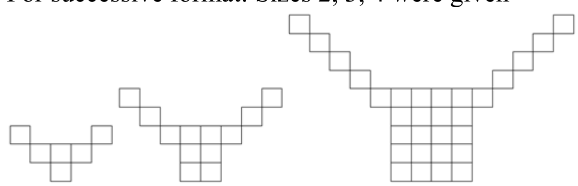
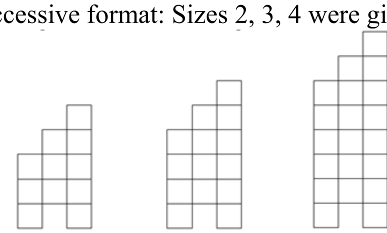
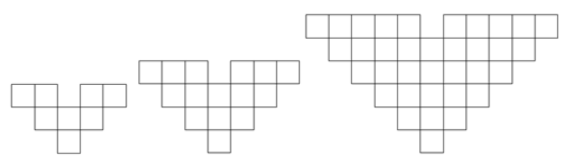
(a) ***Number of generalising tasks*** Deciding on how many tasks to set is a tricky matter: too few tasks may limit the generalisability of the results about the effect of task features on students' success in establishing the functional rule; whilst having too many tasks is simply not practical given the time needed to complete them. After pre-piloting a task to gauge the amount of time students needed to complete it, the decision was to have eight tasks. This number of tasks was believed to be a reasonable figure for covering a range of non-successive configurations.

(b) ***Task scenario*** Most figural generalising tasks used in research and in textbooks ask students to consider a sequence of configurations and then make some near and far generalisations, followed by finding the rule underpinning the pattern depicted in the sequence (see Radford, 2008; Rivera & Becker, 2008). The tasks rarely provide a scenario in which the purpose of representing the pattern with a functional rule might be apparent. For some students, it might, therefore, be difficult to see why they have to do what is required of them. To provide some impetus for students, the notion of purpose and utility was adopted (Ainley, Pratt, & Hansen, 2006) to make the tasks as meaningful as possible for the students. The generalising tasks were framed in different scenarios, such as making wall designs for *Bricks*, and stated the motive as wanting the students to help the character in the task to find the rule for constructing any size (e.g., John wanted to find the number of bricks he had to use to make any size in *Bricks*. Write down the rule John might have used in terms of the size number.).

(c) ***Matching tasks*** To determine whether the type of functions influenced the students' construction of the functional rule, each linear generalising task had a matching quadratic generalising task. Table 3.13 below lists the matching linear and quadratic generalising tasks, with details about the format of pattern display. For each pair of tasks, the description of the scenario was kept invariant: for instance, both *Bricks* (linear) and *Wall Design* (quadratic) were set in the same scenario of creating wall designs using bricks. Furthermore, the shape of the configuration in each linear task was created to resemble as closely as possible that of the matching quadratic task. Considering the Birthday Party Decorations (linear) and *Christmas Party Decorations* (quadratic) tasks for example, both sets of configurations look alike except for the blocks in the middle. Careful considerations to such details during the task design process are believed to be essential as pre-emptive measures for minimising the possible interference of task scenario on the outcome of the *JuStraGen* test so that more robust conclusions can be drawn about the effect of the type of functions on how students construct the functional rule.

Bricks, *Birthday Party Decorations*, *Towers*, *High Chairs*, *Oh Deer!* and *Tulips* were six new generalising tasks designed specially for the *JuStraGen* test. *Christmas Party Decorations* and *Wall Design* were adapted from studies by Rivera (2007), Smith, Hillen and Catania (2007), as well as Warren and Cooper (2008).

Table 3.13: Matching linear and quadratic generalising tasks

Linear	Quadratic
Bricks For successive format: Sizes 1, 2, 3 were given  Size 3	**Wall Design** For successive format: Sizes 1, 2, 3 were given  Size 3
Birthday Party Decorations For successive format: Sizes 1, 2, 3 were given  Size 1 Size 4	**Christmas Party Decorations** For successive format: Sizes 1, 2, 3 were given  Size 1 Size 4
Towers For successive format: Sizes 2, 3, 4 were given  Size 1 Size 2 Size 4	**Oh Deer!** For successive format: Sizes 2, 3, 4 were given  Size 1 Size 2 Size 4
High Chairs For successive format: Sizes 2, 3, 4 were given  Size 2 Size 3 Size 5	**Tulips** For successive format: Sizes 2, 3, 4 were given  Size 2 Size 3 Size 5

(d) ***Parallel tasks*** To determine whether the format of pattern display influenced the students' construction of the functional rule, each task was created in two different formats, with its pattern depicted as (1) a sequence of three successive configurations, and (2) a single configuration or a sequence of two or three non-successive configurations. For instance, Birthday Party Decorations showed three configurations (Sizes 1, 2 and 3) for the successive format and two configurations (Sizes 1 and 4) for the non-successive format.

(e) ***Number of non-successive configurations*** In order for students to move to articulating the functional rule underpinning a pattern, it is noticed from the literature review that there are two common approaches in figural generalising tasks: first, to provide three configurations (see Radford, 2008; Rivera & Becker, 2008; Shaughnessy, 1998; Smith et al., 2007); and second, to show just a single configuration to represent a generic case of the figural pattern, as was discussed previously (Cañadas et al., 2011; Hoyles & Küchemann, 2001; Lannin, 2005; Lannin et al., 2006a; Lin & Yang, 2004; Steele & Johanning, 2004).

What is less common in the literature, however, is the use of two configurations. So far, only three studies have been found using it. The *Ladder* problem in Stacey's (1989) study showed two successive configurations whereas Healy and Hoyles (1999), as well as Warren and Cooper (2008a), used two non-successive configurations in their studies. All these studies provided little, if any, explanation of the rationale for choosing to use these numbers of configurations. But, nonetheless, these numbers do appear to be sufficient to allow students to detect the pattern and then construct the rule. So it can be inferred that having more configurations would not make any difference. Guided by the outcome of the literature review, the present study decided to use one, two or three non-successive configurations in the *JuStraGen* test.

One might now ask whether it is really possible to discern the underlying pattern structure from just a single configuration. To address this concern, it was important to offer a general description of the single configuration. Although the description provided essential information for students to realise how the pattern would grow, it did not disclose the functional rule underpinning the pattern however. Furthermore, the use of a single configuration was limited to only one pair of generalising tasks – *Bricks* and *Wall Design*.

No description of the configuration was given for the remaining pairs of generalising tasks. Like single configuration, the use of two non-successive configurations was also limited to one pair of tasks – *Birthday Party Decorations* and *Christmas Party Decorations*. Three configurations were provided in *Towers* and *Oh Deer!*, as well as in *High Chairs* and *Tulips*.

(f) ***Size number of configurations*** The size numbers of the three given successive configurations ran from either Size1 to Size 3 or Size 2 to Size 4. As for the non-successive format, any single configuration starting from Size 3 was thought to be a reasonable generic case for representing a pattern. Thus Size 3 was given in Bricks and *Wall Design*. Healy and Hoyles (1999), and Warren and Cooper (2008a) used solely odd-numbered sizes (see Figures 2.5 and 2.6(b)) in some of their tasks that involved two or three non-successive configurations. Their choice of configurations might be unfortunate because students might think that the even-numbered sizes did not exist in these tasks. So for generalising tasks with two or three non-successive configurations, it was important to include both odd-numbered and even-numbered sizes so as not to mislead any students into thinking that certain sizes did not exist in the pattern. Therefore, both Sizes 1 and 4 were used in *Birthday Party Decorations* and *Christmas Party Decorations*. In a similar vein, Sizes 1, 2 and 4 were shown in *Towers* and *Oh Deer!*, and Sizes 2, 3 and 5 in *High Chairs* and *Tulips*.

(g) ***Language used*** To help the participating students grasp the task that they were asked to do, all generalising tasks used vocabulary and sentence structures appropriate to their level. Sentences were kept simple and brief, yet still capturing the task's essence. Embedded clauses and complex sentences were also avoided where possible.

(h) ***Structure of task*** All the generalising tasks were unstructured in order to allow students scope for exploration so that they could come up with their own interpretations. This would allow us to see how the students came to recognise and perceive the pattern without scaffolding. So there were no part questions asking for near or far generalisations that would gradually lead students to detect and construct the functional rule underpinning the pattern.

(i) ***Shape of building material*** Square cards or tiles and rectangular bricks were used to build the configurations. Other shapes such as circles and triangles were omitted in order to eliminate the confounding influence of the shape used to build the configurations on students' ability to generalise.

For each pattern format, the eight tasks were divided into two sets of four tasks, administered on two separate days. The task distribution was done in such a way that produced parallel sets of tasks, differing only in pattern format. There were altogether four sets of tasks, labelled as 1S, 1NS, 2S and 2NS. The number in the labels represents the set number of the test and takes either the value 1 or 2. The alphabetical code indicates the format in which the pattern was displayed in the task and assumes the letter S, which stands for successive, or letters NS for non-successive. Set 1S and Set 2S each comprised four generalising tasks (two linear and two quadratic) depicting configurations presented in a successive sequence. Set 1NS and Set 2NS were akin to these two sets except that the configurations were presented in a non-successive manner. Table 3.14 presents an overview of all the eight tasks in the four sets (Sets 1S and 2S, and Sets 1NS and 2NS).

The test duration was 45 minutes for each set and the use of calculators was allowed. Students had to attempt all the generalising tasks in each set. Appendices 3(a) to 3(d) show the four sets of *JuStraGen* test instrument.

The prototype version of the *JuStraGen* test was shown to more than 15 secondary school mathematics teachers attending an in-service workshop on pattern generalisation conducted by the researcher in September 2009, and later by two experts in mathematics education from the National Institute of Education in January 2010 before the pilot study began. The teachers and experts were requested to examine the test instrument to ensure that the generalising tasks met the two objectives of assessing students' ability to generalise and measuring the effect of two task features on their rule construction, and were written with clear instructions and sufficient details. They never raised any major concerns about the tasks except for three suggestions for improving the task instructions. Based on their suggestions, the *JuStraGen* test was modified in the following ways.

(a) A sentence clearly indicating that the pattern grows, rather than repeats, as the terms increase, was inserted into the context of every generalising task. For instance, in *Birthday Party Decorations*, the newly-added sentence was *As the size number became larger, more square cards were used*.

(b) The two instructions asking the participating students for the functional rule and their justifications were reworded in every generalising task as shown below.

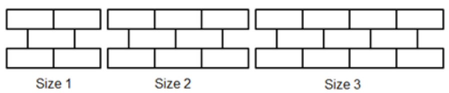
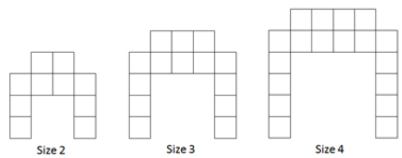
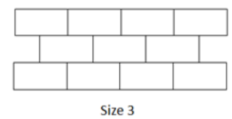
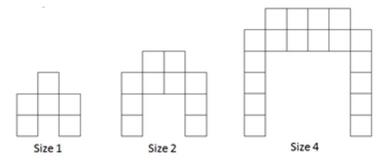
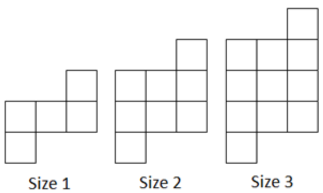
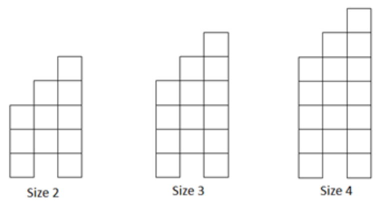
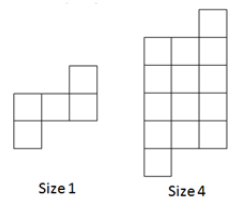
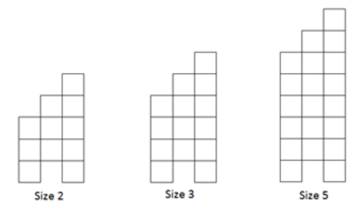
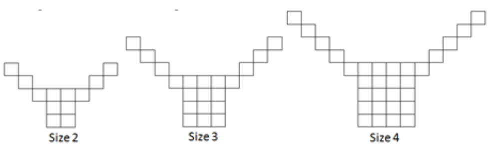
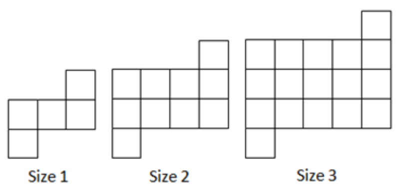
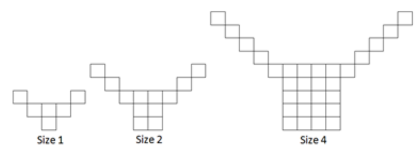
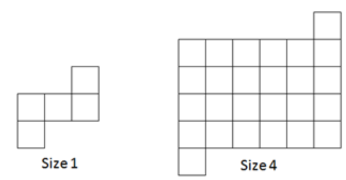
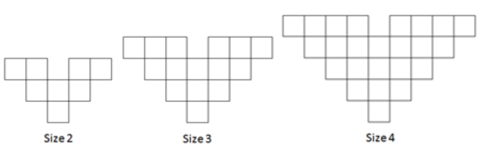
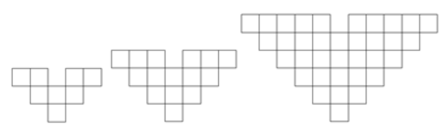
Initial instructions for *Birthday Party Decorations*	Revised instructions for *Birthday Party Decorations*
(a) *Write down the rule Mary might have used.*	(a) *Write down the rule Mary might have used in terms of the size number.*
(b) *Justify how you obtained the rule.*	(b) *Justification: Describe clearly how you obtained the rule. You may use configurations and words.*

This revision of instructions made clear about the kind of rule the participating students were expected to establish and the possible mode of presenting their justifications.

(c) The non-successive configurations in the four generalising tasks were re-arranged and presented in increasing order. Initially, the configurations were given in random order. For instance, the three configurations in *Towers* were displayed in this order: Size 2, Size 5, Size 1. After the revision, their order was Size 1, Size 2 and Size 5. The reason for making this change is to remove the possibility that the order of the configurations could add another confounding factor to the format of pattern display here.

The revised *JuStraGen* test instrument was again shown to the two experts from the National Institute of Education. Both of them were satisfied with the content validity of the test instrument, and so the instrument was ready for piloting.

Table 3.14: Overview of generalising tasks in the four sets

	Successive		Non-successive	
	Set 1S	Set 2S	Set 1NS	Set 2NS
Linear	Q1: Bricks 	Q2: Towers 	Q1: Bricks 	Q2: Towers 
	Q3: Birthday Party Decorations 	Q4: High Chairs 	Q3: Birthday Party Decorations 	Q4: High Chairs 
Quadratic	Q2: Oh Deer! 	Q1: Christmas Party Decorations 	Q2: Oh Deer! 	Q1: Christmas Party Decorations 
	Q4: Tulips 	Q3: Wall Design 	Q4: Tulips 	Q3: Wall Design 

3.1.3.3 Pilot testing of *GAT* and *JuStraGen*

Before the *GAT* and *JuStraGen* instruments were pilot-tested, a pre-pilot trial was conducted in May 2009 with 20 Secondary One students (13-year-old) from a secondary school to gauge the amount of time they would need to formulate and justify an expression for the general term of a pattern in a generalising task that was not used in the *JuStraGen* test. The students were found to take between 10 to 15 minutes to work out and justify the expression for the general term. This means that the students participating in the pilot study would need about an hour and a half to two hours to complete all the eight generalising tasks in the *JuStraGen* test. This test duration was deemed too long for the students after consulting the pilot school's HOD for Mathematics. That was why the eight *JuStraGen* tasks had to be separated into two sets to be administered on two different days, with each set to be completed within one hour.

Over 15 mathematics teachers from different secondary schools and two experts in mathematics education from the National Institute of Education checked the prototype version of the *JuStraGen* test and made a few suggestions to improve the task instructions. After revising the *JuStraGen* test, a pilot study was implemented in School P1 in March 2010 to see how the instrument could be enhanced further, and test all the work involved in its administration. A total of 45 students (29 Express and 16 Normal (Academic)) took the test after school hours on two separate days, which were two days apart. A mathematics teacher from School P1 and the researcher administered the test. The shortest and longest times the students took to complete the test were recorded, and so were any questions asked during the test.

The pilot-testing of the *JuStraGen* test was very helpful to the researcher. It showed that the phrasing and wording of the generalising tasks, as well as the sequence of the tasks in each set of the test were fine. The only minor change made to the test was to enlarge the diagrams in each task. More importantly, the pilot-testing of the *JuStraGen* test also revealed that each set of the test could be completed in about 45 minutes.

The *GAT* instrument was designed and introduced only after the researcher's PhD upgrading examination in November 2010. This explains why it was not trialled together with the *JuStraGen* test in March 2010. An expert in mathematics education from the

National Institute of Education and a retired, experienced HOD for mathematics checked the prototype version of the *GAT* instrument and made a few suggestions to improve the task instructions. The instrument was then piloted with 23 Secondary Two students from School P2 in April 2011 after making amendments to the tasks, with the purpose of testing adequacy of the options of the multiple-choice questions, removing any further ambiguity in the short-answer and structured questions as well as checking the administrative work to be done. The students were given one hour to attempt all the test items. A mathematics trainee teacher, who was taught by the researcher and was then undergoing practicum in School P2, administered the test and recorded any student questions asked during the test. The outcomes of the pilot test showed that the test items were worded explicitly, the number of test items was appropriate, and the test duration was very comfortable as the students were able to complete the test within the given duration.

3.1.3.4 Data collection for main study

Data collection was undertaken from late July to early August 2011 in School M1, mid-August to mid-September 2011 in School M2, and late October 2011 in School M3. Administered under examination conditions and overseen by the schools' mathematics teachers and the researcher, the two sets of the *JuStraGen* test was conducted after school hours on two separate days. The students were given 45 minutes to complete each set of the test and calculators were allowed. Those who completed the test earlier were encouraged to check their answers. Before the test commenced, the students were assured that their performance in the test had no bearing on their academic record, and that their responses would be used solely for research purposes. When the test was over, all test scripts were collected first, then followed by the distribution of the questionnaire for Study II.

3.1.4 THREATS TO INTERNAL VALIDITY OF THE PRESENT RESEARCH DESIGN

The decision to conduct the *JuStraGen* test over two different days could raise some issues as to whether the controlled variable, and not some other unwanted variables, is indeed the one affecting the results. A few potential threats to the credibility and legitimacy of the findings do exist. First, there is a concern that participating students might be conditioned to the demand of the test items after the first test and, consequently, fare better in the second test due to sufficient practice. The next concern relates to student fatigue: that is, the students might become drained and less motivated in the second test than they were in the first, thus affecting their performance in the second test. Finally, there might be experimental attrition: that is, a loss of participating students for reasons, including, withdrawal due to loss of interest, and failure to turn up for at least one test. The loss of students might then change the makeup of Groups 1 and 2. For instance, if a much larger proportion of Group 2 students drops out of the study compared to those dropping out in Group 1, then any difference in student performance might not be attributable solely to the different formats of pattern display.

Counterbalancing, which involves administering the two sets of generalising tasks in different sequences to the two groups of students, was considered initially to mitigate especially the first two concerns. For instance, Group 1 students could receive Set 1 of the *JuStraGen* test on the first day and Set 2 on the second day whereas Group 2 students could be assigned Set 2 first then followed by Set 1. As the two sets of generalising tasks would not be administered to each group of students within the same day, students from both groups might have the opportunity to compare and discuss the various tasks they did in the respective sets. For this reason, the idea to use counterbalancing was abandoned after much deliberation to prevent any possibility of leakage of questions.

As can be seen in Table 3.15, the four generalising tasks in Set 1 differ from those in Set 2 not only in the shape of the configurations and task scenario, but also in other ways. For instance, the three successive configurations of the linear tasks were deliberately chosen to start from Size 1 to Size 3 in Set 1S and from Size 2 to Size 4 in Set 2S. The same consideration was similarly done for the quadratic tasks in the two sets. As for the two corresponding sets with non-successive configurations, the linear tasks in Set 1NS and in Set 2NS had different formats of pattern display, so did the quadratic tasks as well. Therefore, the two sets of generalising tasks, whether Sets 1S and 2S or Sets 1NS and 2NS, could be assumed as somewhat independent. With this assumption, the possibility that student performance in the second test was due to their experience in the first test was reduced.

3.2 STUDY II: AN EXPLORATION OF WHAT STUDENTS BELIEVE IS THE BEST-HELP GENERALISING STRATEGY

The second part of the present study aimed to explore what the participating students would judge as the most helpful generalising strategy for establishing an expression for the general term of a pattern, and whether they could work out the expression for the general term using the best-help strategy of their choice. The remaining section begins with a discussion of the research design for Study II, followed by a description of the research instruments, participating students and research method.

3.2.1 RESEARCH DESIGN

One of the few techniques available to provide information on beliefs is survey. Since one of the two objectives of Study II was to gather data about student belief of a best-help strategy for constructing a functional rule for a pattern, a survey research design was particularly appropriate and thus adopted. The survey method is especially efficient for "collecting data in large amounts…in a short period of time" (Burns, 2000, p. 568), and this strength of the survey method fitted the research design of Study I very well. First, the survey could be conducted immediately after the *JuStraGen* test in Study I, and second, it could involve all the participating students rather than a sample of these students. Survey data were collected by means of a questionnaire containing closed items, in which the students were required to select a strategy that was judged as most helpful from an array of alternatives. A detailed description of the questionnaire is provided in Section 3.2.3.1 below.

The attempt to use closed items in the survey questionnaire to gather data could lead to students simply guessing and selecting a random strategy from the array of alternatives. This weakness of closed items in the questionnaire was overcome by interviewing students face-to-face after carrying out the survey to probe their capability to produce a correct functional rule using their chosen strategy. Doing the student interviews had the potential to provide the needed confirmation of the efficacy of the student choice of best-help strategy as well as to increase confidence in the data interpretation. Alongside these two advantages, probing the students could prevent and reduce the case of "don't know" and non-responses as the researcher could press for explanation and clarification. Crucially, the student interviews also permitted further investigation of any student's failed attempt to construct a functional rule using the chosen best-help strategy. The unsuccessful student had to re-examine the alternatives provided in the *QBBS* question, re-choose the best-help strategy, and then re-formulate the rule, this time using the newly selected strategy. The interviews were expected to be very time-consuming and, therefore, only about 5% of the student sample in the main study would be picked for the interviews due to time and manpower considerations.

All the student interviews were recorded using a tape recorder and a camcorder. The voice recording was to capture the dialogue between the researcher and the students verbatim and the video recording to film the students' written responses on their respective questionnaires whilst they were working out the functional rule. The recordings also freed the researcher up to observe the students and participate in the dialogue, rather than having to focus on note-taking.

This part of the research study on the investigation of the efficacy of the student choice of best-help strategy was only introduced following feedback from the upgrading examiners regarding a couple of rather bold and ambitious research objectives in the original plan for Study II. After much rethinking and deliberation, Study II was then reconceptualised and the interview segment was added.

Two *QBBS* tasks, *Birthday Party Decorations* and *Christmas Party Decorations*, were selected for the interview segment. For each task, a student from each course was picked for each of the four alternative methods provided in the task from amongst those who had judged that particular method as most helpful. Thus eight Express and another eight Normal (Academic) students were chosen for the interview. Each of these 16 students was asked to develop the functional rule for only one task using the generalising strategy that he or she had judged as most helpful.

3.2.2 SUBJECTS

This section is aimed at detailing the profiles of participating students and their respective schools in both the pilot study and the main study for Study II.

3.2.2.1 Pilot study

The 45 students, 29 Express and 16 Normal (Academic), from School P1 who took part in the pilot-testing of the *JuStraGen* test instrument in March 2010, also participated in the trialling of the *QBBS* questionnaire, which was the main test instrument for Study II. The student profiles were already described earlier in Section 3.1.2.1.

The interview schedule was trialled in School M1 in mid-August 2011, following the administration of the *JuStragen* test for the main study. The pilot-testing of the interview schedule could not begin earlier in another school not involved in the main study due to administrative constraints in seeking an appropriate school with students comparable to those in the main study. Two students, a boy and a girl from the Express course, were picked by the researcher for the trial based on their *JuStraGen* responses and choice of best-help strategies in the questionnaire.

3.2.2.2 Main study

The survey in Study II was carried out on the same 515 students, comprising 337 Express and 178 Normal (Academic), as in Study I. The profiles of these students were detailed in Section 3.1.2.2 above.

For the student interviews, the selection of the students was based on their performance in the *JuStraGen* test and their choice of best-help strategies in the *QBBS* survey. From amongst the 515 students, the researcher identified 16 (3%) students, with eight each in the Express and Normal (Academic) courses. These students, eight girls and eight boys, comprised five from School M1, seven from School M2 and four from School M3. The ratio of interviewees from Schools M1, M2 and M3 was initially 3 : 3 : 2 respectively according to the proportion of students from these schools in the entire student sample. But due to the poorer test performance of students from School M2, the researcher decided to interview more students from School M2 by reducing the apportioned number of students from School M1, resulting eventually in the final ratio of 5 : 7 : 4. The interviews were conducted in the respective schools over a period of four months from August to November 2011, with a break in early and mid-October because of the schools' end-of-year examinations. The interviews were then resumed in the last week of October after the school examinations. Table 3.15 below summarises the profiles of the students involved in the interviews and the interview duration. For instance, a girl, coded 64M3, judged *repeated substitution* in *Birthday Party Decorations* as the most helpful generalising strategy.

Table 3.15: Profiles of interviewed students and interview duration

Generalising strategies in *QBBS* tasks	Express		Normal (Academic)			
	BD	Xmas	BD	Xmas		
Repeated substitution	64M3, G	77M3, B	217M1, G	136M2, G		
Constructive	15M1, G	44M2, G	148M2, B	138M3, B		
Reconstructive	37M1, B	75M2, B	160M2, G	141M2, B		
Deconstructive	11M2, G		153M3, B			
Figure-ground Reversal		93M1, G		149M1, B		
Schools	Girls	Boys	Girls	Boys	Total	Interview Duration
M1	2	1	1	1	5	Mid-Aug – Sep 2011
M2	2	1	2	2	7	Mid-Sep – Nov 2011 (interruption in Oct for school examinations)
M3	1	1		2	4	Last week of Oct 2011
Total	5	3	3	5	16	

BD: Birthday Party Decorations; Xmas: Christmas Party Decorations; G: girls; B: boys

3.2.3 INSTRUMENTS

The data for Study II was collected using two research instruments: a *Questionnaire of Students' Belief of Best-help Strategies* (*QBBS*), and an *Interview Schedule* (*IS*). The next two sections describe these instruments in detail.

3.2.3.1 QUESTIONNAIRE OF STUDENTS' BELIEF OF BEST-HELP STRATEGIES (*QBBS*)

The *QBBS* instrument offers a survey of secondary school students' belief of what they would judge as the most helpful generalising strategy for establishing the functional rule underpinning a pattern. It is a questionnaire that consists of four questions designed to explore what generalising strategy would best help the students in rule construction. The four questions were divided into two separate sets, each comprising two questions. The first set, labelled as *QBBS*1, contained the two tasks *Birthday Party Decorations* and *Bricks* from Set 1S of the *JuStraGen* test instrument, and the second, labelled as *QBBS*2, contained *Christmas Party Decorations* and *High Chairs* from Set 2S.

In the construction of questionnaires, three types of items are used generally: closed items, open-ended items, and scale items (Burns, 2000). Of the three item types, the present study decided on the closed items, which invited the participating students to pick a response from a few alternatives. With fixed alternatives, such items have the advantage of achieving greater uniformity of measurement and of being more easily coded than the open-ended items. On the other hand, disadvantages included the possibility of subjecting the participating students to unnecessary stress with too many alternatives and of annoying those who might find all the alternatives unsuitable.

Set in a context of a discussion amongst four students, each question provided four possible student solutions, from which the participating students had to choose one. Each student solution represented a different way of constructing the rule based on the classification scheme for generalising strategies widely reported in the research literature (see Section 2.3.2). In each solution, the three configurations given in the questions were used only to illustrate how the underpinning pattern could be interpreted. There was no disclosure of the functional rule that could be abstracted from such an interpretation. The participating students were asked to pick the student solution that they judged would best help them to construct the functional rule. A description of the student solutions from all four questions in *QBBS* is presented below.

Consider the four student solutions in *Birthday Party Decorations* from *QBBS*1. Method 1 employs what Rivera and Becker (2008) called a *constructive* strategy (S1) when the original configurations are viewed as being created from three non-overlapping columns of cards. Method 2 uses a *numerical* strategy (S2) known as the repeated substitution strategy (Bezuszka & Kenney, 2008). In this student solution, the number of cards used in the next consecutive figure is found using that of the preceding one. In Method 3, each original configuration is visualised as being made up of identical components that overlap (S3). The generalising strategy used is what Rivera and Becker (2008) termed as *deconstructive* strategy. Method 4 involves rearranging the original configurations into something that students find more familiar (S4). Chua and Hoyles (2010a) called such a generalising strategy as *reconstructive* strategy. For *Bricks* also from *QBBS*1, Methods 1, 2, 3 and 4 correspond to S2, S4, S1 and S3 respectively.

For *Christmas Party Decorations* from *QBBS*2, Methods 2, 3 and 4 correspond to S1, S4 and S2 respectively. In Method 1 which illustrated the *figure-ground reversal* strategy (S5), the original configurations are augmented to become part of a larger rectangle with cards missing in the top and bottom rows. For *High Chairs* also from *QBBS*2, Methods 1, 2, 3 and 4 correspond respectively to S4, S5, S2 and S1.

The *QBBS* instrument was designed on the basis of the following considerations:

(a) ***Choice of generalising tasks*** All eight generalising tasks in the *JuStraGen* test were tested with two batches of in-service mathematics teachers from 2008 to 2009. These teachers were participants of an in-service workshop on pattern generalisation conducted by the researcher. They were asked on the last day of the workshop to work out the functional rule for each task in two different ways using any generalising strategies that they were not only familiar with, but would also use in their teaching as well. Their solutions and choices of strategies were examined carefully. Tasks that supported the use of a wide range of different strategies were then identified. Amongst these tasks, *Birthday Party Decorations*, *Bricks*, *Christmas Party Decorations* and *High Chairs* were eventually selected.

Birthday Party Decorations, *Bricks*, *Christmas Party Decorations* and *High Chairs* were selected due to certain resemblance amongst the configurations in these tasks. Both *Birthday Party Decorations* and *Bricks* are isomorphic tasks. In other words, the configurations share not only the same structural build-up, but also the same underlying pattern. With identical features in these two tasks, one would expect the students to pick the same method that is most helpful to them in establishing the rule. These two questions, therefore, provide checks for consistency of the students' choices between them. *Birthday Party Decorations* and *Christmas Party Decorations* are matching tasks. The structural build-up of their configurations bears some close resemblance, except that the rule for *Birthday Party Decorations* is linear whereas that for *Christmas Party Decorations* is quadratic. *High Chairs* was chosen because its configurations have underlying structure that is somewhat similar to those of *Birthday Party Decorations*. Thus, *High Chairs* and *Christmas Party Decorations* are useful and appropriate for comparing students' choices between a linear task and a quadratic task.

(b) ***Number of questions*** Since each set of *QBBS* was supposed to immediately follow each *JuStraGen* test, the number of questions per set should be kept low but enough to produce the desired information, and at the same time, manageable for students to complete both the test and questionnaire under an hour. The time allocated was ample, yet not too long for the students especially when they had to stay back after school to take part in this study. So it was decided that four questions were sufficient for *QBBS*. The four questions were split into two questions per set.

(c) ***Type of generalising strategies*** When dealing with figural generalising tasks, it is important that students are able to link the visual representation of the functional rule with its symbolic representation (Noss, Healy, & Hoyles, 1997). In other words, they must be able to recognise the pattern structure underpinning the given configurations and abstract the functional rule directly from these configurations. With this point in mind, the student solutions provided in every *QBBS* question were developed to include more figural solutions than numerical solutions.

The type of generalising strategies to be represented in the student solutions was determined from the in-service teachers' solutions for the *JuStraGen* tasks. Of those numerical solutions observed, the *repeated substitution* strategy was the teachers' top preference. As for the figural solutions, the *constructive* and *reconstructive* strategies were their clear favourites. Thus, all these three strategies were featured in every *QBBS* question. For the less popular figural strategies, a decision was made to include the one that involved viewing the configurations as being made up of overlapping components in *QBBS*1, and the one that involved augmenting the original configurations in *QBBS*2. Both strategies were picked based on the pilot students' responses in the *JuStraGen* test, in which a few of them applied the former strategy in *Birthday Party Decorations* whilst some used the latter strategy in both *Christmas Party Decorations* and *High Chairs*. To sum up, each *QBBS* question offered four student solutions, each representing a different generalising strategy.

In the researcher's opinion, the number of student solutions per *QBBS* question was a viable figure for covering a decent variety of generalising strategies.

Clearly, the four student solutions did not encompass every possibility. For instance, the *Birthday Party Decorations* configurations could be envisioned using the *figure-ground reversal* strategy. But this alternative was not offered for two reasons. First, the strategy was not observed amongst the in-service teachers' solutions, so its use in that generalising task might be rare. Second, the researcher did not intend to offer too many alternatives to cloud the participating students' thinking and stress them out when they were caught in a dilemma over which strategy to choose from. On the other hand, giving too few alternatives was equally unrealistic as it might restrict the students' choices of strategies, much to their annoyance.

(d) ***Sequencing of student solutions*** The four generalising strategies represented in the student solutions were sequenced in different ways from question to question in each set of *QBBS*. For instance, in *QBBS*1, the order of generalising strategies for Methods 1 to 4 was S1, S2, S3 and S4 respectively in *Birthday Party Decorations* whereas it was S2, S4, S1 and S3 in *Bricks*. This is a way to safeguard the accuracy and reliability of the data collected. Thus it is reasonable to consider a student who picked Method 1 in *Birthday Party Decorations* and Method 3 in *Bricks* as someone who really found S1 most helpful.

A prototype version of the *QBBS* was passed to one expert in mathematics education from the National Institute of Education and an experienced secondary school mathematics teacher for checking the face validity and content validity in January 2010. They suggested making minor revisions such as rewording some parts of the discussion that took place amongst the four students and enlarging the diagrams given in each task in the questionnaire. Following these revisions, the second version of *QBBS* emerged. It was then passed to a group of in-service mathematics teachers attending a workshop on algebra in April 2010 to get their feedback. Other than expressing a concern that students might find the method using S3, which appeared only in *QBBS*1, somewhat hard to follow, the teachers were of the view that *QBBS* did not need any further revision. Appendices 4(a) and 4(b) show the two sets *QBBS* instruments.

3.2.3.2 Interview Schedule (*IS*)

The aims of the student interviews were (a) to clarify the students' rules in the *JuStraGen* test, (b) to illuminate their thinking process and generalising strategies used in formulating the rules, (c) to determine the efficacy of their choice of best-help generalising strategy selected in the *QBBS* survey, and (d) to unfold their reasons for choosing or rejecting a particular strategy offered in the *QBBS* questionnaire. The interview questions are listed in Table 3.16 below.

Table 3.16: Questions used in the interview

Purpose	Questions
To explore students' understanding of the term *rule*	1. What do you think the question is asking you to find? 2. What do you understand by the word *rule*?
To explore students' understanding of the variable used	For students who produced rule such as (n + difference): 3. What does n in your rule represent?
To allow students to discover their rules are incorrect	Let's take a look at your rule. Your rule is (*read out the rule written in JuStraGen Test*). 4. Can you tell me the number of cards (tiles) in Size 3 using your rule? Size 5? 5. Is the number the same as that given in the diagram?
To guide students to derive the functional rule	6. Can you tell me the number of cards (tiles) in Size 10? Size 50? Size 100? (Choose smaller size number such as 4 or 5 if the student is facing difficulty with Size 10.) 7. How do you know if you are correct? 8. How did you figure out what Size (missing term) would look like? 9. What is the number of card (tiles) in Size n? 10. How did you come up with this rule? 11. Which part of the given diagrams makes you notice this pattern?
To encourage students to read aloud their thoughts when they slip into silence for too long	12. Tell me what you are thinking about now? 13. Can you tell me what difficulty you have in this question?
To ask students to generate a functional rule using the strategy they judged as most helpful	Let's take a look at the survey question. 14. Can you recall which method you chose as the most helpful approach to construct the rule? 15. Imagine you are (name of character), tell me how you would form the rule, based on the chosen method.
To probe the students' reasons for choosing or rejecting a particular strategy	16. Why did you choose Method (number)? 17. Why did you NOT choose Method (number)? 18. If a student's strategy in the *JuStraGen* Test is different from the best-help strategy, ask: Comparing the two methods (place the JuStraGen solution and best-help strategy side-by-side), which method will best help you to obtain the rule? Why?

Each interview began by seeking clarifications on the student's rule and generalising strategy for *Birthday Party Decorations* or *Christmas Party Decorations*, depending on which task he or she was supposed to develop the rule for. Following that, the *QBBS* task was shown and the student was invited to recall his or her choice of best-help strategy first before being asked to use it to develop the functional rule. When any student slipped into silence whilst he or she was developing the rule, verbal cues were offered to remind him or her to read aloud his or her thoughts. Additionally, the interview also probed the students' justifications for favouring a particular strategy over the other three. The insights gained would help to interpret the findings of this research study. Moreover, whenever time permitted, clarifications were sought on the student's rule and generalising strategy in other *JuStraGen* tasks. Each interview, lasting between half an hour to45 minutes, was later transcribed for data analysis.

3.2.3.3 Pilot testing of *QBBS* and *IS*

The *QBBS* questionnaire, which was the main test instrument for Study II, was pilot-tested together with the *JuStraGen* test instrument in March 2010 over two separate days with 45 students (29 Express and 16 Normal (Academic)) from School P1. Trialling the questionnaire was essential to reveal any confusing and problematic description of the various alternative approaches that still existed in the tasks. Items that were covered with the students pilot-testing the questionnaire included checking i) any words that were unfamiliar, 2) the clarity of the task wording, 3) the format of the tasks, and 4) the actual time required to complete the questionnaire. The students were informed that blank slip of paper was available for them to suggest other alternative approaches should they find any of those provided unsuitable.

The pilot study found that the sequence of the *QBBS* tasks was fine, the task wording and format were clear, and the students were able to complete each set of the questionnaire within 10 minutes. So no changes were made to the questionnaire.

Two Express students from School M1 were interviewed in mid-August 2011 using the *IS*. The aims were to familiarise the researcher with the questions, and to allow him to find a good position to place the tape recorder and camcorder. Following the trial interviews, changes were made to the wording of some questions, a few questions were removed and replaced with relevant ones. Through the pilot testing, the researcher realised that some students needed more time to articulate their thinking, to work out the functional rule using their chosen strategy, and to justify their choice of best-help strategy. As a result, these students might not be able to handle too many tasks within a reasonable timeframe. Finally, the pilot testing of the *IS* helped the researcher not only to develop some experience and confidence in conducting an interview with the students, but also to identify an appropriate spot for positioning the tape recorder and camcorder.

To sum up, the pilot testing of *QBBS* and *IS* showed that each set of the questionnaire could be completed comfortably within 15 minutes, and each student interview could take about 30 to 40 minutes.

3.2.3.4 Data collection for main study

Data collection for the *QBBS* survey was undertaken concurrently with the *JuStraGen* test from late July to early August 2011 in School M1, mid-August to mid-September 2011 in School M2, and late October 2011 in School M3. After the test was completed, all the test scripts were collected before the *QBBS* questionnaires were distributed to the students. The students were given 15 minutes to complete each set of the survey.

The student interviews were conducted from mid-August to September 2011 in School M1, mid-September to mid-November 2011 in School M2, and late October 2011 in School M3. The students were interviewed individually by the researcher. Before the interview commenced, every student was informed that the interview would be audio-recorded and video-taped and it was, therefore, very important for him or her to speak his or her thoughts clearly and loudly. The student was also assured that his or her performance in the interview had no bearing on his or her academic record, and that his or her identity would be protected. During the interview, the student was shown his or her *JuStraGen* test script and *QBBS* questionnaire, on which he or she was asked to write any additional workings and the functional rule developed using the best-help strategy.

3.3 CONFIDENTIALITY, ETHICS APPROVAL AND CONSENT

Both student and school confidentiality and anonymity are of the utmost importance when implementing the present study. The anonymity of the participating students and schools was maintained by assigning each student and school a unique identification code during the data analysis. As mentioned previously, the students were identified by the code NMn, where N refers to their identification number and Mn their school code when reporting about them individually in this manuscript. As well, the schools were identified as School M1, School M2 and School M3 when writing about them.

Ethics approval and consent were obtained from three organisations. First, the present study was subject to ethics review by the researcher’s supervisor and a member of the researcher’s advisory committee in November 2009 before data collection started. Ethics approval was granted and the outcome was then reported to the Institute of Education’s Research Ethics Coordinator. Second, a request for approval to collect data from schools was submitted to the Ministry of Education of Singapore in early July 2011. Approval was granted around mid-July 2011. Lastly, a letter of consent cum information sheet (see Appendix 1) was given to each participating school before the main study started in late July 2011. The letter provided detailed information about the present study and sought the schools’ consent to allow their students to partake in the study. This letter was not issued to the students’ parents as the schools were able to act *in loco parentis*. The HODs for Mathematics from all the three schools gave their consent.

3.4 DEVELOPMENT OF ANALYTIC RUBRIC AND CODING SCHEMES

This section describes how student responses in the *JuStraGen* test were being scored, and how their rules, the modality of their rule, their choice of generalising strategies and of justification schemes were being coded.

3.4.1 RUBRIC FOR SCORING *JUSTRAGEN* TEST

A holistic rubric is a rating scale that scores a student response as a whole product without assessing the separate components (Arter & Chappuis, 2007; Nitko & Brookhart, 2011). In contrast, an analytic rubric examines the separate components of the student response and scores each component separately on a numerical scale first, then sums up the individual numerical scores to obtain a total score (Nitko & Brookhart, 2011). An advantage of using a holistic rubric is the speed of the scoring process. Scoring is quicker than when using an analytic rubric because the student response is read through to just get an overall impression of what the student has accomplished.

Despite the fact that an analytic rubric is more time consuming to apply than a holistic one, the present study still decided to use it to score student responses in the *JuStraGen* test for the reason that its use was in alignment with the key objectives of this study. As two of the key objectives concerned students' rule construction and their use of generalising strategies, the student responses should then be scored according to these two criteria rather than based on an overall impression of the responses, which was what a holistic rubric would score. Thus an analytic rubric was developed based on the responses of the pilot students. The initial analytic rubric comprised two criteria: (1) *Rule construction* and (2) *Use of generalising strategies*. Each criterion was a six-point scale from 0 to 5.

With over 500 students in the main study and each of them had to do eight generalising tasks in the *JuStraGen* test, the maximum possible number of responses to be scored would be more than 4000. For such a large number of responses, it is reasonable to anticipate a more diverse range of student responses. As such the initial analytic rubric might not be adequate enough for assessing all the *JuStraGen* responses in the main study. Thus a review of the initial analytic rubric for relevance was necessary. Eventually it underwent a few cycles of revision in the following ways until it stablised.

(a) Cycle 1: The initial analytic rubric was applied to a sample of student responses from Set 1 of *JuStraGen* test. These responses came from four intact classes in Schools M1 and M2. Two classes, one in each course, were selected from each school. Around 140 of all the 515 scripts of the *JuStraGen* test were scored. A new descriptor was created for any of the two criteria if student responses did not fit any of the descriptors. A few descriptors were refined to improve clarity of expectations required in the student responses. In this way, a fine-grained analytic rubric, referred to as the second version of the analytic rubric, was produced.

(b) Cycle 2: The second version of the analytic rubric was subsequently applied to another sample of student responses, this time coming from one Express and one Normal Academic classes in School M3. Over 70 scripts were scored using the second version of the analytic rubric. The second version still needed further modification and addition in the rule construction criterion to accommodate the varied student responses. So the third version of the analytic rubric emerged.

(c) Cycle 3: The third version of the analytic rubric was subsequently applied to a third sample of student responses, coming from a class in each course in Schools M1 and M2. Nearly 120 scripts were scored using the third version of the analytic rubric. This time, the rubric was found to be more stable, with only minor rewording of a couple of descriptors in both criteria. So the fourth version of the analytic rubric eventually emerged from a cyclic process of adding and refining. Subsequently, it was used to rescore all the 515 scripts of the *JuStraGen* test.

The fourth version of the analytic rubric is found in Appendix 5. Briefly, for each criterion, a score of 0 to 2 points was assigned to a response that did not link the number of bricks, cards or tiles in each configuration with its size number. Those that did earned 3 to 5 points. Responses that gave a recursive rule, although correctly described the underpinning pattern, were not awarded a full score of 5 points. This is because such a rule was not expressed in terms of the size number, which all the generalising tasks specifically asked for. Thus stating a recursive rule was not deemed a completely correct response to justify a full score. A brief illustration of this scoring rubric using the *Bricks* task is presented in Table 3.17 below.

Under *use of generalising strategy*, using pattern spotting to correctly establish the functional rule earned 3 points, which is the lowest possible score that could be awarded for showing a structural relationship. Further, student responses showing a correct functional rule obtained from an unidentifiable generalising strategy also earned the lowest possible score of 3 points.

To show how the fourth version of the analytic rubric was applied, seven examples of student responses for the *Bricks* task and one example for the *Oh Deer!* task (see Figure 3.9) were provided below. These examples illustrate the different scores for the *rule construction* and *use of generalising strategy* criteria. The participating students were labelled as NMn, with N referring to the student's identification number assigned by the researcher and Mn the school identification code. So Student 25M3 is a student from School M3 whose identification number is 25. RC and GS refer to the *rule construction* and *use of generalising strategy* criteria respectively.

Table 3.17: Analytic scoring rubric for Bricks

Criteria	Score	Descriptor
Rule construction	5	giving a correct functional rule (e.g., $5 + 3(n - 1)$)
	4	giving (a) an incorrect rule due to minor slips in algebraic manipulation (e.g., $5 + 3(n - 1) = 5 + 3n - 1 = 3n + 4$), or (b) a workable rule but not expressed in terms of the size number (e.g., $3n - 1$, where n is the number of bricks in the top row)
	3	using a functional relationship to show the structure of configurations without clearly stating the functional rule (e.g., $5 + 3(10 - 1)$ for Size 10)
	2	giving a correct recursive rule (e.g., add 3 to get the next term)
	1	(a) expressing the recursive rule incorrectly in algebraic notation (e.g., $n + 3$), or (b) stating the first differences correctly without stating the recursive rule, or (c) showing how a particular shape was obtained
	0	giving (a) a partially correct answer without leading to any recursive or functional rule (e.g., the top and bottom rows are equal but one more than the middle row), or (b) an incorrect or blank answer.
Use of generalising strategy	5	showing clear evidence of using numerical or visual cues from the pattern to derive the correct general rule
	4	working out the structure of non-immediate terms
	3	working out the structure of immediate terms
	2	using the differences between terms to obtain the recursive rule
	1	describing the nature of the terms
	0	using an incorrect strategy or giving a blank response

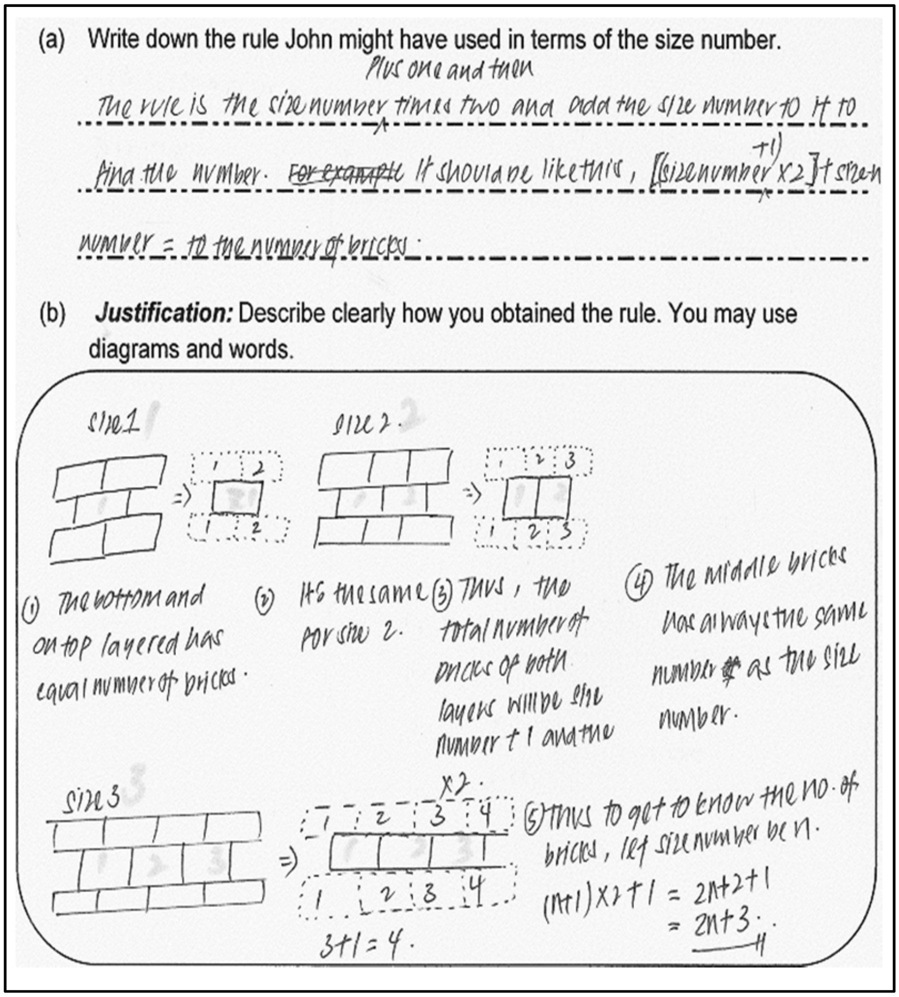
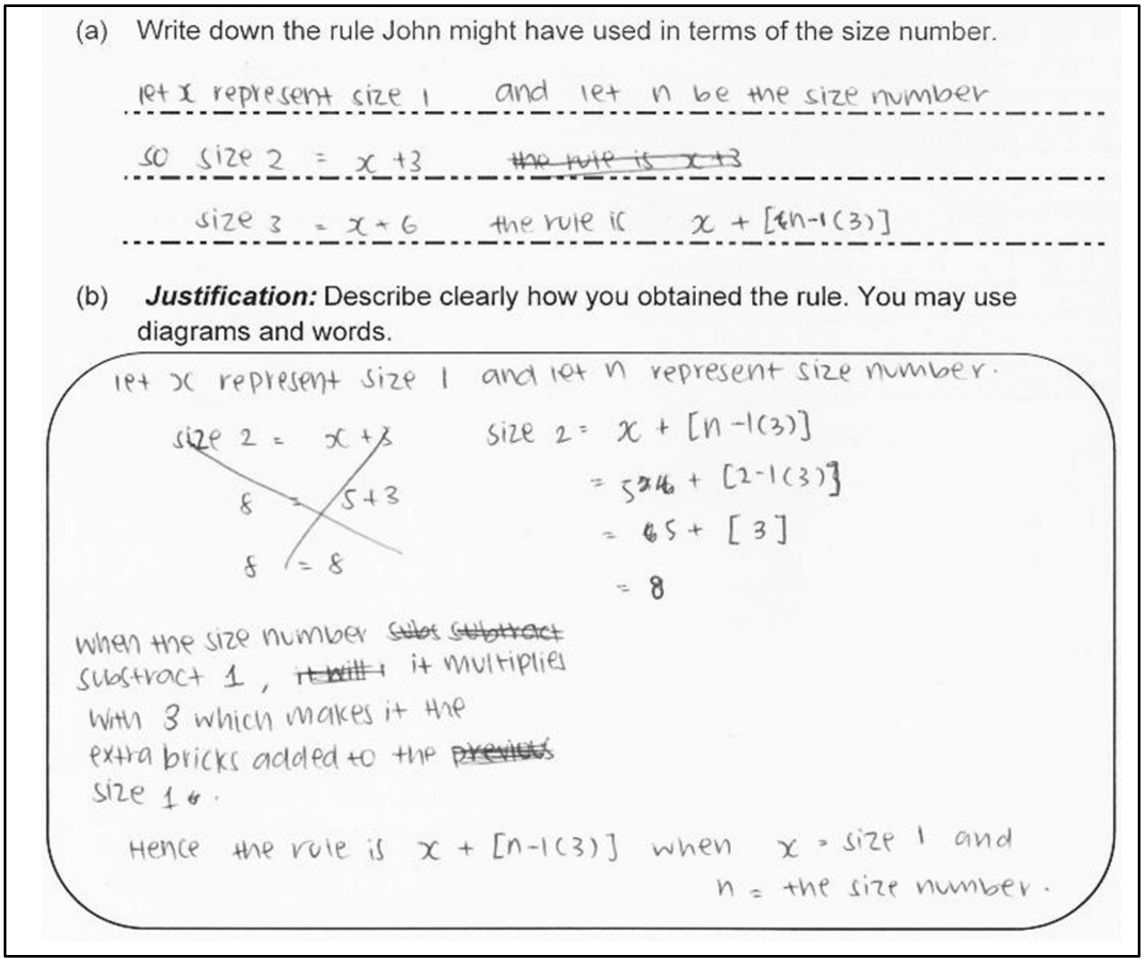


Figure 3.4. Bricks response of Student 20M3

Figure 3.4 shows a Group 1 student's response for the Bricks task that obtained a full score of 10 points. As clearly indicated in part (a), Student 20M3 constructed a correct functional rule, $[(size\ number + 1) \times 2] + size\ number$, using a correct generalising strategy that was evident in the justification provided in part (b). Such a rule was accepted as correct even if it was not expressed in the closed form of $3 \times size\ number + 2$. Further, it can be noted under justification that the student made a mistake when expressing the rule in algebraic notation (see Point 5). However, the mistake occurred only in the final step of the justification and was not carried through from part (a). For this reason, this mistake was not taken into consideration when scoring the response. Thus 5 points were given each for RC and GS.



(a) Write down the rule John might have used in terms of the size number.

let x represent size 1 and let n be the size number

so size 2 = $x + 3$ ~~the rule is $x+3$~~

size 3 = $x + 6$ the rule is $x + [n-1(3)]$

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

let x represent size 1 and let n represent size number.

size 2 = $x + 3$

8 = 5 + 3

8 = 8

size 2 = $x + [n-1(3)]$

= 5 + [2-1(3)]

= 5 + [3]

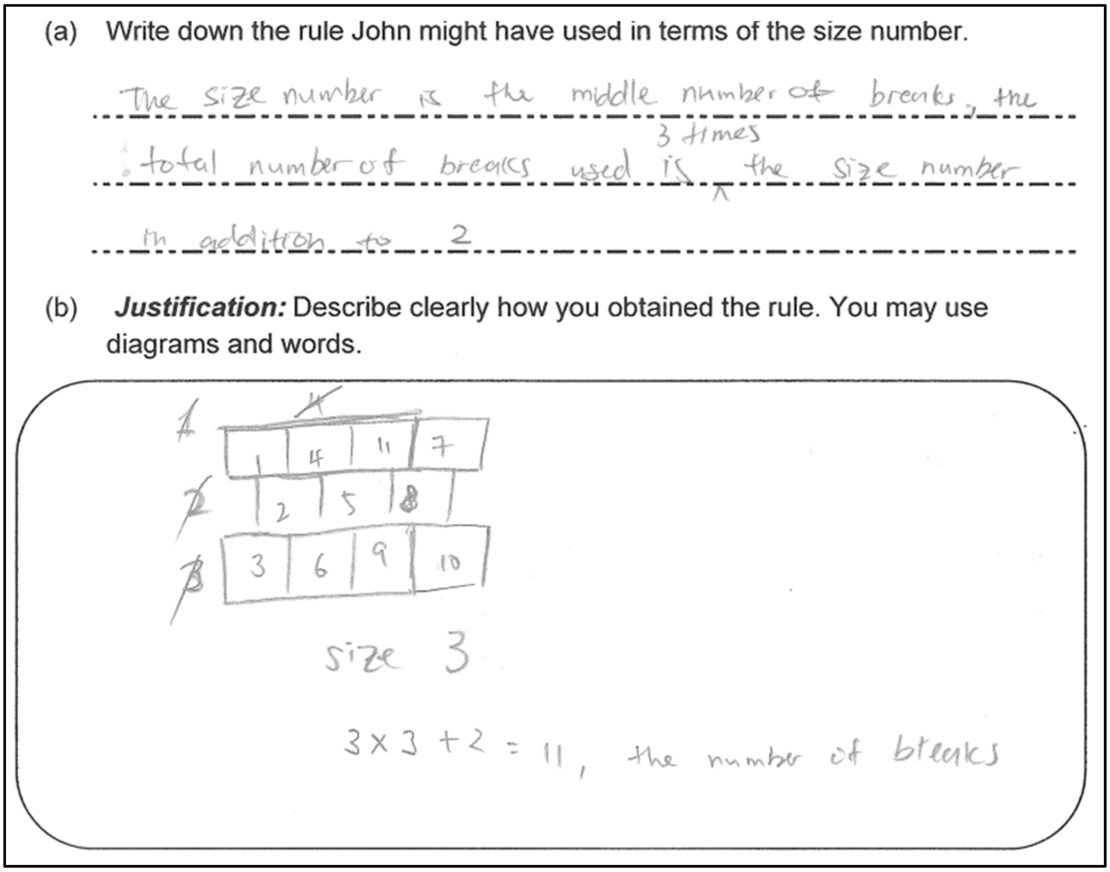
= 8

when the size number ~~subs subtract~~ substract 1, ~~it will~~ it multiplies with 3 which makes it the extra bricks added to the ~~previous~~ size 1.

Hence the rule is $x + [n-1(3)]$ when x = size 1 and n = the size number.

Figure 3.5. Bricks response of Student 23M2

Figure 3.5 presents a Group 1 student's response that scored 9 points. To find the number of bricks needed in Size *n*, Student 23M2 figured out that $3(n-1)$ bricks must be added to Size 1 using an inductive approach, as can be seen in the working (that is, add three bricks to Size 1 to get Size 2, add six bricks to Size 1 to get Size 3). However, the general expression representing the additional bricks to be added was incorrectly given as $n-1(3)$. The functional rule was, therefore, wrong. From the working of finding the number of bricks in Size 2, it was clear that $n-1(3)$ actually meant $3(n-1)$. So in view of the correct reasoning, the mistake was treated as a minor slip. This was why 4 points were given for RC and 5 points for GS.



(a) Write down the rule John might have used in terms of the size number.

The size number is the middle number of breaks, the total number of breaks used is 3 times the size number in addition to 2

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Figure 3.6. Bricks response of Student 18M3

Figure 3.6 shows a 8-point response from a Group 2 student. Although the functional rule was correct, Student 18M3 did not describe clearly how it was derived. Thus this response scored 5 points for RC and 3 points for GS.

Figure 3.7 below shows a 7-point response given by a Group 2 student. Splitting the configuration into three separate rows, Student 97M1 was able to express the number of bricks in each row in terms of the size number. The only problem with this response was that the three parts were not combined to form the functional rule. So 3 points were given for RC. For illustrating how a non-immediate configuration (Size 6, when given only Size 3) was obtained, the response scored 4 points for GS.

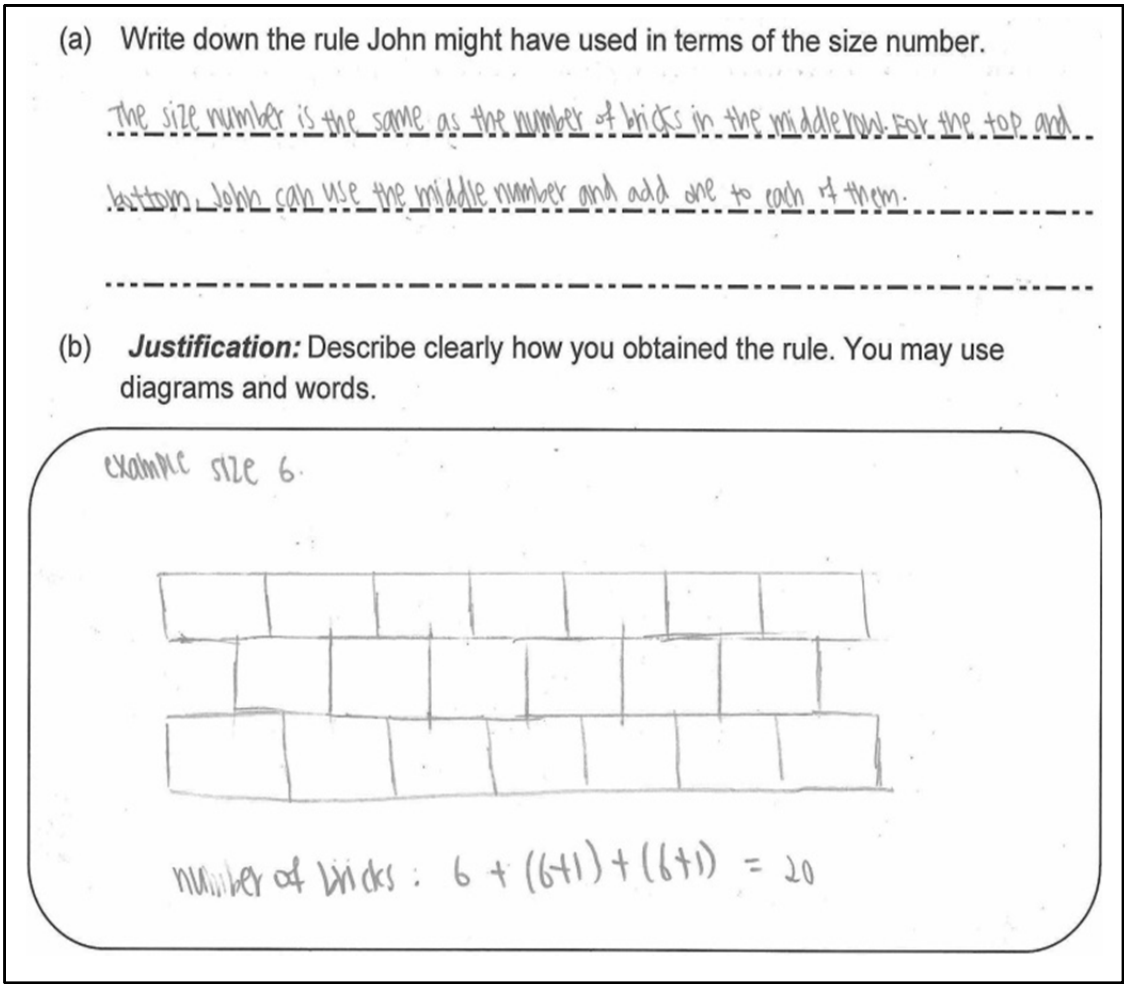


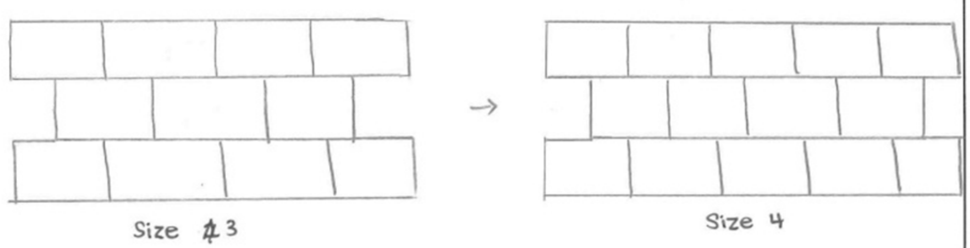
Figure 3.7. Bricks response of Student 97M1

The two examples of Group 2 students' responses in Figure 3.8 below were given 4 points. In (a), Student 85M2 stated clearly the recursive rule, supporting with a diagram suggesting three bricks were being added to Size 3 to build Size 4. So both RC and GS each scored 2 points. Unlike in (a), the response in (b) by Student 69M2 did not offer any recursive rule. What it presented instead was the generalising strategy used to build the configurations from Size 1 to Size 3. Since this response explained the formation of immediate configurations when provided with only Size 3, RC gained 1 point whilst GS gained 3 points.

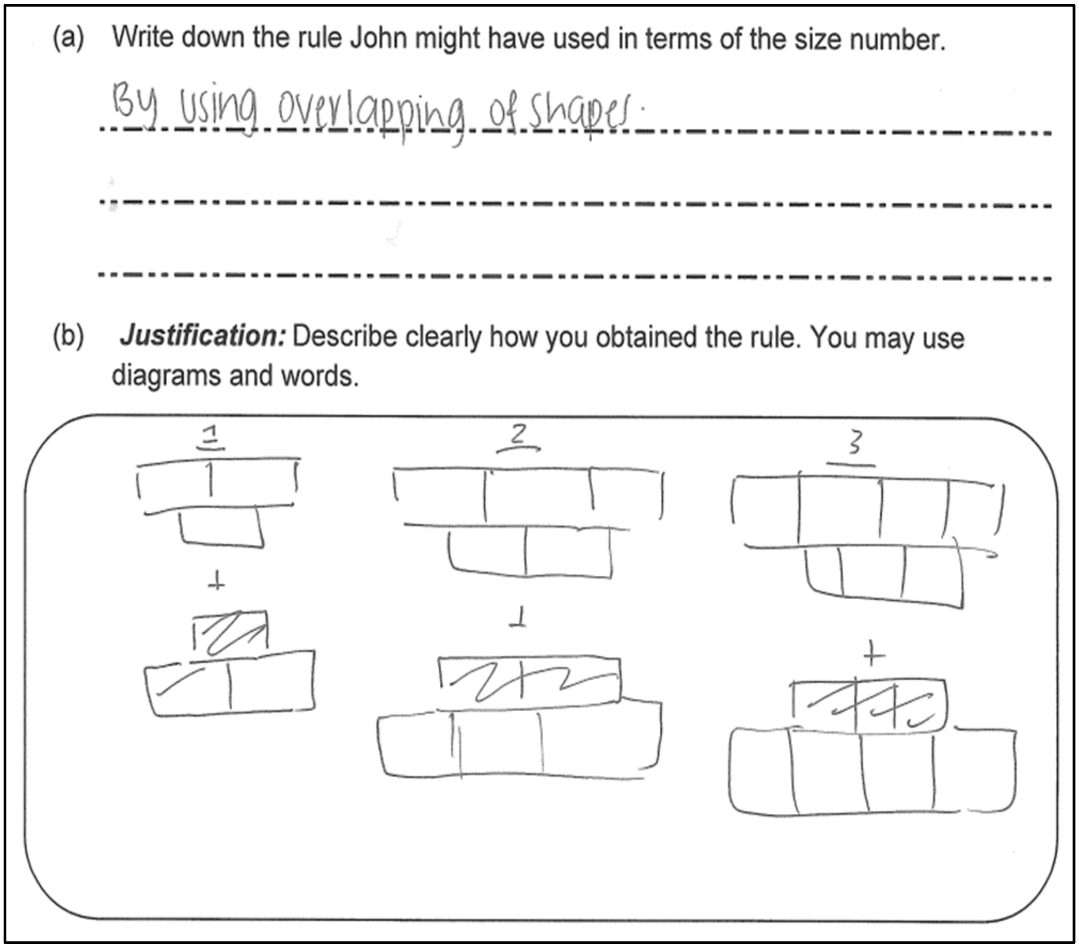
(a) Write down the rule John might have used in terms of the size number.

The rule is
Increase the number of bricks by 3 with every size increased.

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.



(a) Student 85M2


(a) Write down the rule John might have used in terms of the size number.

By using overlapping of shapes.

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

(b) Student 69M2

Figure 3.8. Bricks responses of two students

Figure 3.9 shows a Group 1 student's response for the *Oh Deer!* task that scored 1 point. Student 109M3 counted the number of bricks in Size 2, Size 3 and Size 4 and noticed that the three terms were all even numbers. This act of noticing then prompted the student to give the rule as *only even numbers could be used*. Such a response expressed neither a recursive rule nor a functional rule for the underpinning pattern, hence RC scored 0 point. But for describing the nature of the three terms, GS scored 1 point.

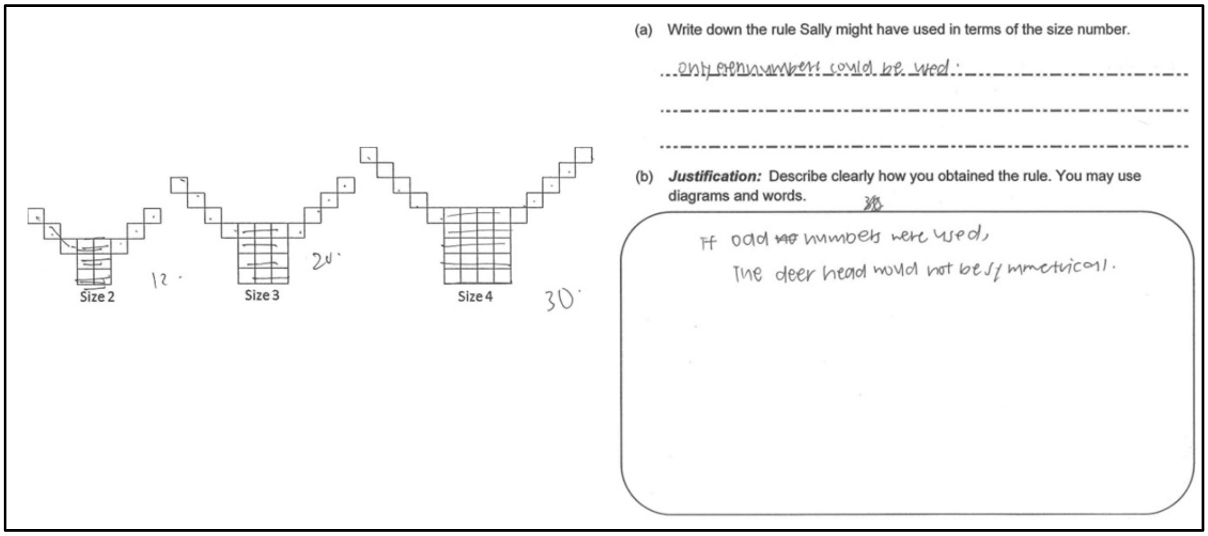


Figure 3.9. Oh Deer! response of Student 109M3

Figure 3.10 presents a Group 2 student's response that scored 0 point. Student 171M2 noted correctly that both top and bottom rows of the given Size 3 configuration were identical and each had one more brick than the middle row. Apart from this observation, nowhere in the response did the student mention the constant difference between any consecutive configurations or the relationship between the number of bricks in the middle row and the size number. So this response, although not wrong in general, seemed to merely describe the outward appearance of the Size 3 configuration as seen in the diagram. Whether the student knew how to construct this configuration from the previous configuration or when given its size number remained uncertain. Rather than regarding the response as a description showing how Size 3 was constructed, it was considered as partially correct but not leading to any rule, therefore scoring 0 point for both RC and GS.

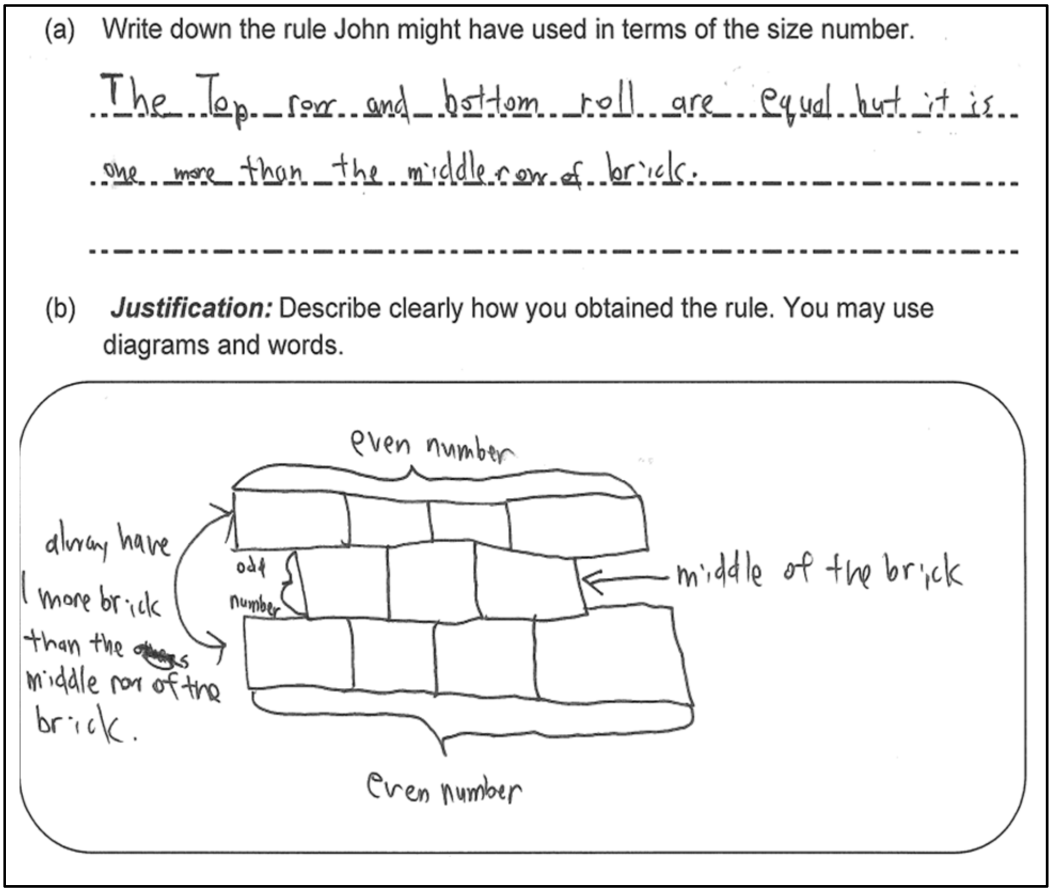


Figure 3.10. Bricks response of Student 171M2

3.4.2 THE CODING SCHEME FOR THE TYPE OF RULES AND FOR THEIR MODALITY

Coding is a process of combing the data that researchers have collected to look for data that share some common characteristics, and then assigning a code label to these similar data so that they can be easily retrieved later for further investigation (Lodico et al., 2010). A coding scheme is a framework designed to organise and categorise those similar data for comparison, analysis, interpretation and drawing of conclusions. It consists of the different code labels used to mark those data that share related themes.

In the present study, the initial coding scheme for the type of rules was developed using *a priori* ideas drawn from the results from the pilot study, the initial analytic rubric, the solutions of the in-service teachers who checked the *JuStraGen* test instrument, as well as the researcher's solutions using the various generalising strategies previously reported in Section 2.3.2. For each of the eight generalising tasks in the *JuStraGen* test, all the correct equivalent functional rules constructed by the students in the pilot study were collated and examined. When two or more equivalent expressions of the functional rule were seen in a student response, the one that captured how the pattern structure was visualised was coded. Each different rule thus formed a category. A three-digit code was then assigned to each category. Take the *Bricks* task for instance. The two correct but different looking rules, $3n + 2$ and $2(n + 1) + n$, were classified under two separate categories with different codes. In contrast, equivalent recursive rules for each of the four linear generalising tasks were interpreted as similar. So these rules came under the same category, which was also assigned a three-digit code. That is, rules such as *increase by 3* and $T_n = T_{n-1} + 3$ belonged to the same category. The reason for classifying equivalent recursive and functional rules differently was because all the generalising tasks asked for the functional rule and a key objective of this present study was to determine the different types of functional rules that the students were capable of establishing.

For the case of quadratic generalising tasks, it was anticipated that students with a tendency to derive a recursive rule would experience significant difficulties in expressing such a rule. This is because quadratic generalising tasks were not as common as linear generalising tasks in the local secondary school mathematics textbooks. Thus the students would not likely be familiar with articulating the recursive rule for quadratic patterns correctly. But they should be able to identify the first differences between two consecutive configurations, just like what they would do in a linear generalising task. So listing the first few terms of the first differences as a sequence seemed a plausible way to articulate the recursive rule. However, those who did it this way might be merely stating the differences between any consecutive configurations given in the generalising tasks, without noticing how the first differences were growing as the size number increased. Such an act of noticing the difference beyond the first differences might not be obvious to the students because, in Mason's (2002) words, noticing was not something they did deliberately. To distinguish those students who were clear about that from those who did not, another recursive rule to watch out for was the one that explained how the next term of the first differences was determined from the previous term. Of the two recursive rules, the latter was deemed a more complete description for two reasons. First, to explain how each term of the first differences was obtained from the previous term, the students had to examine the relationship between the consecutive terms to see their differences. When doing this, they should then notice that the second differences were constant and not growing with the size number like the first differences. This act of noticing was particularly crucial as it led to the second reason. That is, recognising the second differences allowed the students to find the next term of the first differences from the previous term. This, in turn, permitted them to extend the quadratic pattern beyond the given terms. Since quadratic generalising tasks are not so common in the Singapore secondary school mathematics curriculum, it would definitely be interesting to see how the participating students described the quadratic recursive rule. For this purpose, the two ways of articulating the quadratic recursive rule were not classified under the same category but kept as separate categories.

This initial coding scheme was applied to all four classes from School M3. Slightly over 150 of all the 515 scripts of the *JuStraGen* test were coded. Every response was matched with the available codes. New but similar rules were subsumed under the same category. For instance, when the functional rule for the *Bricks* task, $2n + (n + 2)$, was first encountered, it was regarded as similar to an existing code for the rule, $2n + (n + 1) + 1$. So this code was expanded to comprise the two rules. For quadratic generalising tasks, it was rather unanticipated to find some students like Student 4M1 and Student 33M1 noticing that the first differences depicted a linear relationship and expressed these differences as an algebraic rule. Student 4M1 produced the response *Sally might have used* $T_{n-1} + (2n + 2)$ for the *Bricks* task whilst Student 33M1 gave the response *Tony might have added* $2n + 1$ *to the previous size to find the size* for the *Tulips* task. Their correct rules were also added to the category of recursive rule in the respective generalising task. If a correct functional rule did not match any of the available codes, a new code was created. In this way, a second version of the coding scheme emerged. It was found during the coding process that neither the inclusion of rules into the existing codes nor the addition of new codes would affect the previous coding of responses. So the second version of the coding scheme was subsequently applied to all the 515 scripts of the *JuStraGen* test, also continuing to add similar rules to available codes and create new codes along the way. Appendices 6(a) to 6(h) provide the final version of the coding schemes for the types of rules in all eight tasks.

The coding pattern for the three-digit code was derived as follows:

(a) The leftmost digit, if it is 1 to 8, indicated the task number in the *JuStraGen* test. Digit 1 for *Bricks*, 2 for *Oh Deer*!, 3 for *Birthday Party Decorations*, 4 for *Tulips*, 5 for *Christmas Party Decorations*, 6 for *Towers*, 7 for *Wall Designs* and 8 for *High Chairs*.
(b) The middle and rightmost-digits, if it is 01 to 19, were used to code equivalent functional rules.
(c) The middle and rightmost-digits, if it is 20 or 21, were used to code recursive rules. Code 20 was for responses showing how the term of the first differences was determined from its previous term. Code 21, only available in all the four quadratic generalising tasks, was for responses that listed some terms of the first differences without any explanation of how they were obtained.
(d) A workable rule not expressed in terms of the size number was coded as 950.
(e) A recursive rule that was incorrectly expressed in algebraic notation was coded as 960.
(f) A partially correct response that does not lead to any rule or a description of how a particular configuration was obtained was coded as 970.
(g) An incorrect response was coded as 980.
(h) A blank response was coded as 990.

The initial coding scheme for the modality of the rule was simpler and more straightforward to develop. From the pilot study, it was found that the recursive rule was typically expressed in words. On the other hand, the functional rules were provided in one of the following three modes of representation:

(a) Completely in words: e.g., *size add one and multiply it to size add two*
(b) Completely in algebraic notations: e.g., $(n + 1)(n + 2)$
(c) In alphanumeric form: e.g., $(size\ number + 1)(\text{size number} + 2)$

Other than producing a recursive or functional rule, some student responses were partially correct, some described only particular cases, some were totally incorrect or irrelevant whilst some were left blank. So the initial coding scheme for the modality of the rule was constructed based on these findings. A single-digit code was used for each category in the following manner:

(a) A rule that was expressed completely in words was coded as 1.
(b) A rule that was expressed completely in algebraic notations was coded as 2.
(c) A rule that was expressed in a alphanumeric form was coded as 3.
(d) A partially correct rule was coded as 4.
(e) A rule that described particular cases was coded as 5.
(f) A totally incorrect or irrelevant rule was coded as 6.
(g) A blank response was coded as 9.

The initial coding scheme for the modality of rule was applied concurrently with the initial coding scheme for the type of rule to over 150 scripts of the *JuStraGen* test. The *description-of-particular-cases* category (Code 5) was created originally to account for those student responses that showed how particular cases were obtained. However, due to a low occurrence of such responses for each task, it was decided to merge this category with the *partially correct rule* category (Code 4), which accounted for responses that were incomplete but could possibly lead to a correct functional rule if done fully. The modality of the rule was finally classified according to one of the following six categories:

(a) A rule that was expressed completely in words was coded as 1.
(b) A rule that was expressed completely in algebraic notations was coded as 2.
(c) A rule that was expressed in alphanumeric form was coded as 3.
(d) A partially correct rule or a description of particular cases was coded as 4.
(e) A totally incorrect or irrelevant rule was coded as 5.
(f) A blank response was coded as 9.

Subsequently, all the 515 scripts of the *JuStraGen* test were recoded using the revised coding scheme, which is found in Appendix 7. The *JuStraGen* responses presented in the previous section on the analytic scoring rubric are cited below to illustrate how the type of rule and the modality of rule were being coded.

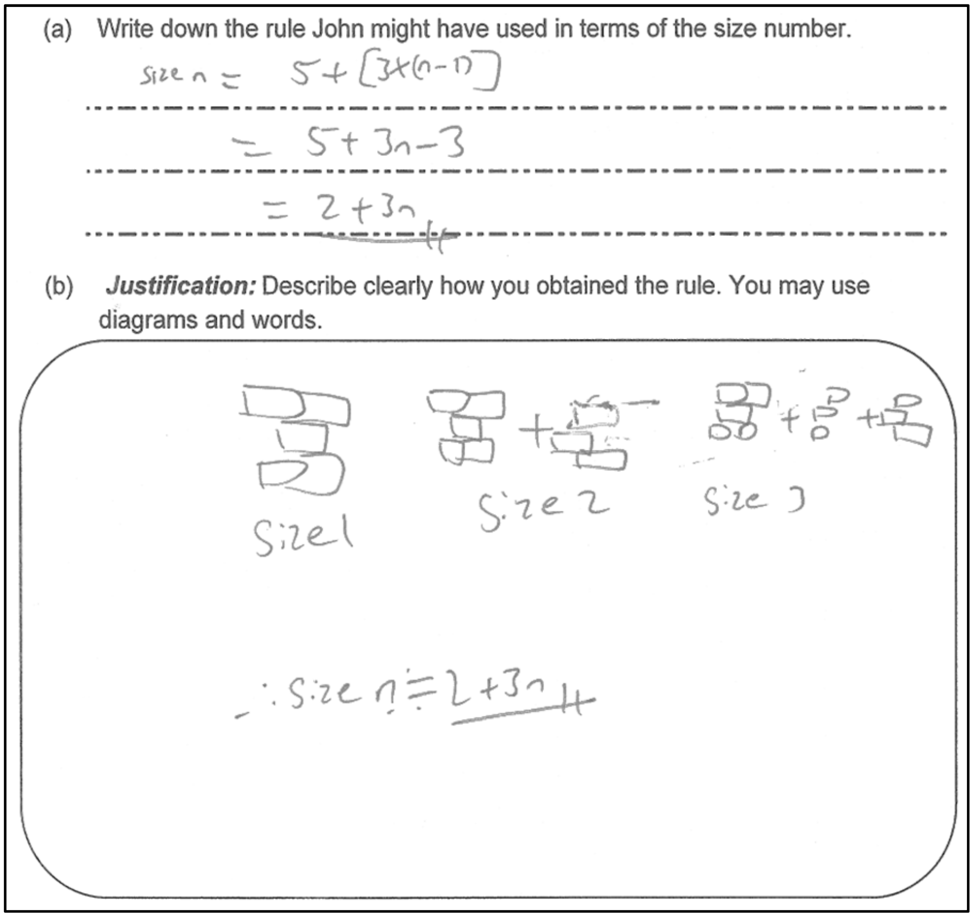
In Figure 3.4, Student 20M3 expressed the same functional rule first in words, then in alphanumeric form. Later, the student's attempt to use letters to express the rule was unsuccessful. The rule, $[(size\ number + 1) \times 2] + size\ number$, was coded as 103 for the type of rule and 3 for the modality of rule. The reason for choosing to code the modality of this rule over the one expressed in words was because the former rule was deemed a more sophisticated response than the latter. The ability of using symbols to articulate the rule is a key aspect of the learning of pattern generalisation in schools.

The response of Student 18M3 in Figure 3.6 is an example of a rule expressed completely in words: *the total number of breaks used is 3 times the size number in addition to 2* (sic). The rule was coded as 101 for the type of rule and 1 for the modality of rule.

As previously explained, although the functional rule of Student 23M2 in Figure 3.5 was not exactly correct, it was very clear that the intended rule was actually $5 + 3(n - 1)$. So the student's rule was still coded as 102 for the type of rule and 2 for the modality of rule. Thus, depending on the root of the mistake, an incorrect rule was not always treated as wrong and coded 980 for the type of rule.

For partially complete responses such as those of Students 97M1, 69M2 and 171M2 (see Figures 3.7, 3.8(b) and 3.10 respectively), they were assigned the code 970 for the type of rule, and the code 4 for the modality of rule. Finally, the response of Student 109M3 in Figure 3.9 was completely wrong, hence coded 980 for the type of rule and 5 for the modality of rule.

When a student response showed two correct equivalent rules where the first was simplified to the second, the first rule was coded for the type of rule and its modality. For instance, in Figure 3.11 below, Student 2M3 first produced $5 + 3 \times (n - 1)$, then simplified it to $2 + 3n$. So the former expression, and not the latter, was coded.



(a) Write down the rule John might have used in terms of the size number.

size n = 5 + [3×(n−1)]

= 5 + 3n − 3

= 2 + 3n

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Figure 3.11. Bricks response of Student 2M3

3.4.3 THE CODING SCHEME FOR GENERALISING STRATEGIES

The generalising strategies in the initial coding scheme were identified from a range of sources, including, mainly the research literature (e.g., Becker & Rivera, 2005; Bezuszka & Kenney, 2008), the results from the pilot study, and the researcher’s observations made during the scoring of the student responses in the *JuStraGen* test. The strategies were classified broadly into four categories: (1) using only numerical cues established from the pattern, (2) using only visual cues established directly from the structure of the configurations, (3) using guess-and-check, and (4) miscellaneous such as using an incorrect strategy or using an indeterminate strategy. The first category comprised strategies such as finding only the differences between consecutive terms, expressing the next terms using the immediate term preceding them, and substituting values into the formula for linear rule.

The second category included the *constructive*, *deconstructive* and *reconstructive* strategies and the third category comprised the *guess-and-check* strategy. Student responses that showed a correct rule derived from an indeterminate strategy or that were partially complete were classified under the last category. A two-digit code was assigned to each strategy.

The coding pattern for the two-digit code was derived as follows:

(a) The leftmost digit indicated the category as described above. Digit 1 for the first category, 2 for the second, 3 for the third and 4 for the last.
(b) The rightmost digit was used to code the different strategy within the category. For instance, *constructive*, *deconstructive* and *reconstructive* strategies in the second category were coded as 21, 22 and 23 respectively.
(c) A blank response was coded as 99.

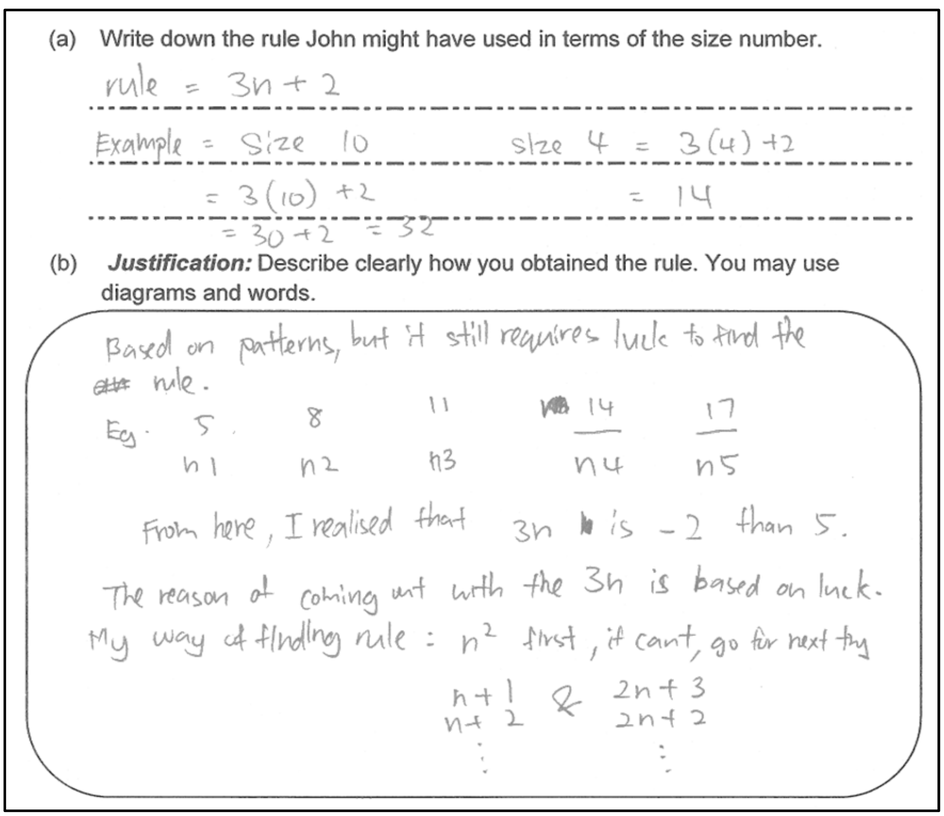
Some students, however, used a combination of two strategies to derive the functional rule. In this thesis, the plan combining two different strategies for constructing a rule is called a *combo* strategy. A four-digit code was devised for a *combo* strategy, with the two leftmost digits representing the code of the first generalising strategy used whilst the last two denoting the code of the second strategy used. For instance, the code 2123 was assigned to a student response that manifested the application of the *constructive* strategy first, then followed by the *reconstructive* strategy.

Over 150 scripts of the *JuStraGen* test were coded using the initial coding scheme for the generalising strategy, carried out concurrently with the initial coding schemes for the type of rule and the modality of rule. In each task, the generalising strategy of every student was matched with the available codes; otherwise, a new code was created. The code 2123 was a good example of a new code developed after the strategy was not found to match any of the available codes. In this way, a second version of the coding scheme, found in Appendix 8, then emerged and was subsequently applied to all the 515 scripts of the *JuStraGen* test. To give readers a better sense of how the generalising strategies were being coded, some *JuStraGen* responses presented in Section 3.3.1 above are illustrated below.

Student 23M2, in Figure 3.5, discerned Size 2 in the *Bricks* task as adding three bricks to the five in Size 1, Size 3 as adding six bricks (two times of three) to Size 1, and hence, by induction, Size n as adding $(n - 1)$ times of three bricks to Size 1. The student's strategy was to express the number of bricks in each size in terms of that in the preceding size, and the number of bricks in the preceding size was, in turn, further expressed in terms of the number of bricks in Size 1. Such a generalising strategy was coded as 12.

As the response in Figure 3.7 clearly shows, Student 97M1 noticed that the number of bricks in the middle row was the same as the size number, and that the top and bottom rows each had one brick more than the middle row. This way of viewing the configurations as being constructed from three non-overlapping rows of bricks is a manifestation of the *constructive* strategy. Hence this response was coded as 21 for generalising strategy used.

Guess-and-check is a fairly common strategy that students used to obtain the rule. Students' use of this strategy is manifested in two ways: by own admission or by clear evidence in their justifications. Consider the *Bricks* response in Figure 3.12(a) below.


(a) Write down the rule John might have used in terms of the size number.

rule = $3n + 2$

Example = Size 10 = $3(10) + 2$ = $30 + 2$ = 32

size 4 = $3(4) + 2$ = 14

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Based on patterns, but it still requires luck to find the rule.

Eg. 5, 8, 11, 14, 17

n1, n2, n3, n4, n5

From here, I realised that $3n$ is -2 than 5.

The reason of coming out with the $3n$ is based on luck.

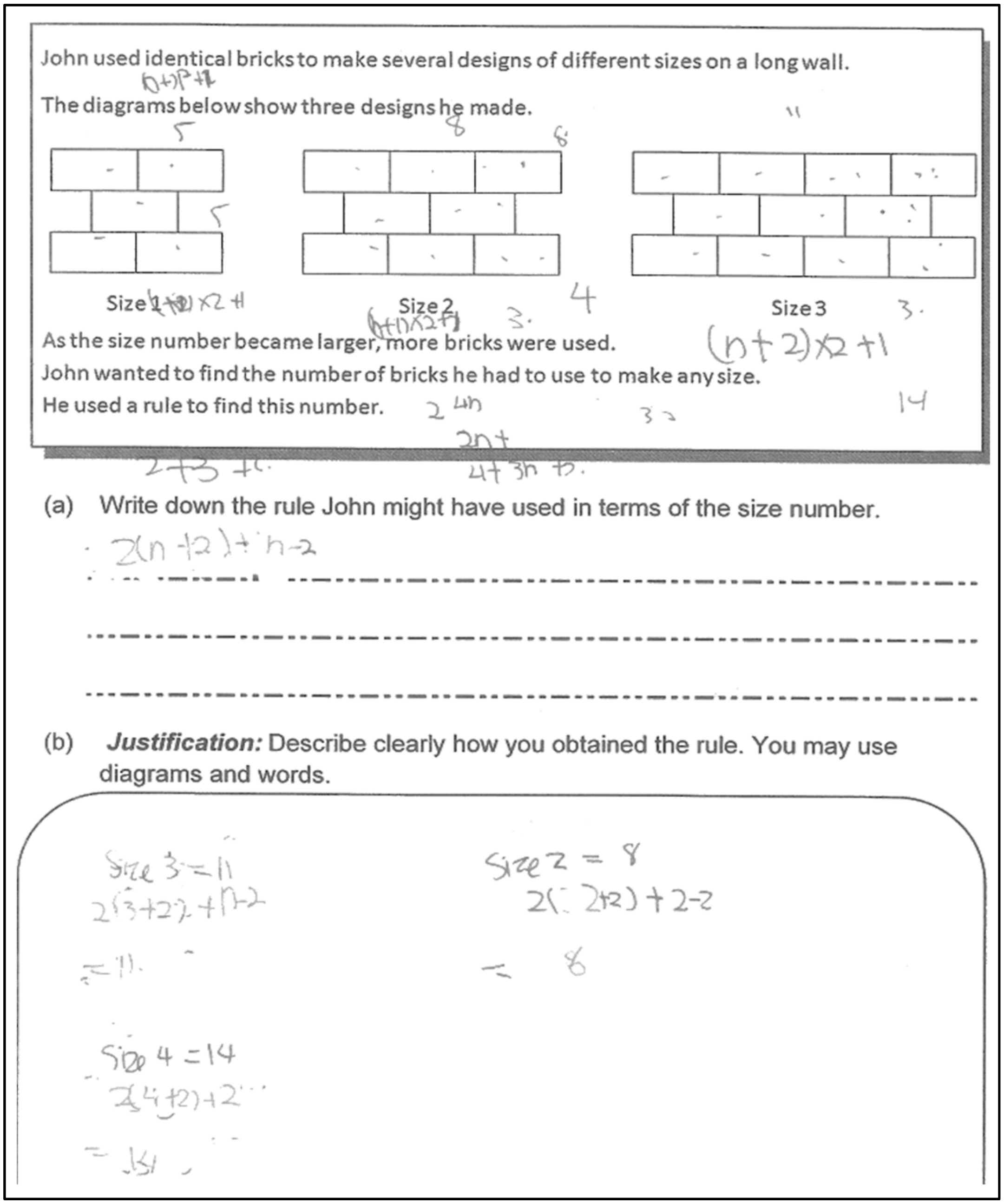
My way of finding rule: n^2 first, if can't, go for next try

$n + 1$ & $2n + 3$

$n + 2$ $2n + 2$

⋮ ⋮

(a) Student 16M3



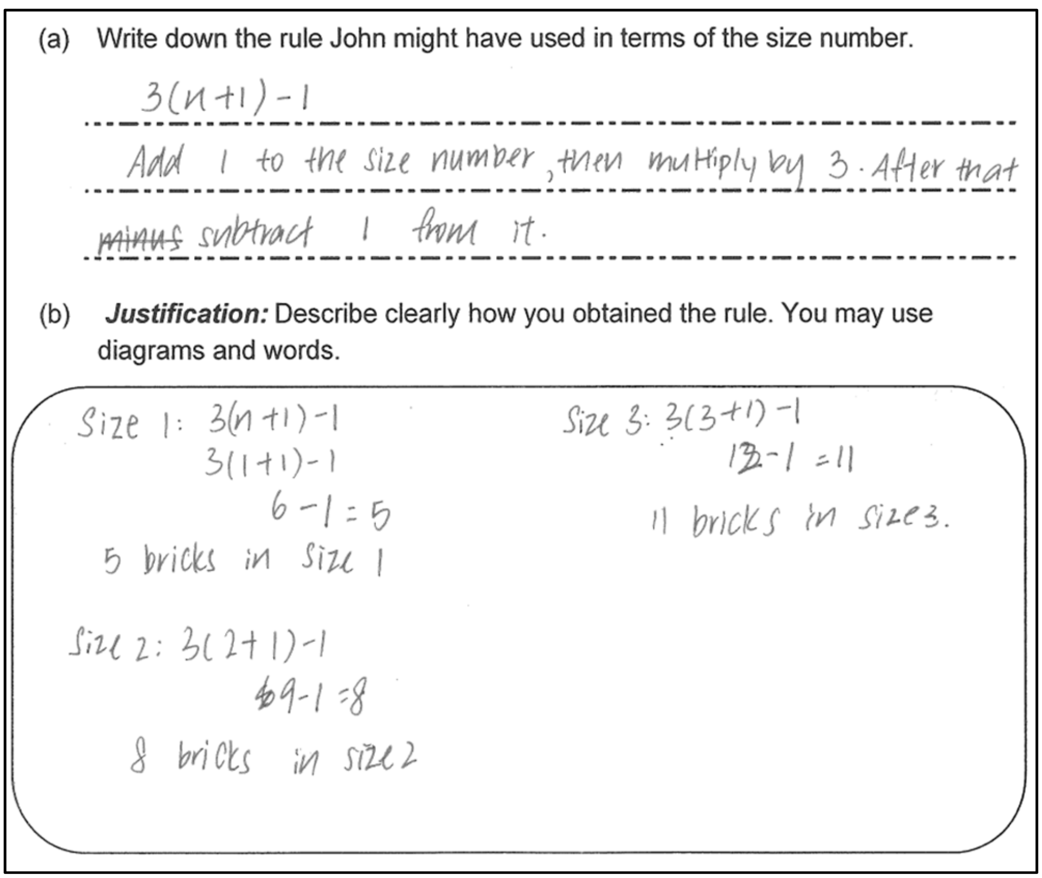
(b) Student 69M3

Figure 3.12. Bricks responses using guess-and-check

Student 16M3 somewhat spotted that 3*n* was consistently two fewer than the number of bricks in Figure *n* through guessing and checking of a few rules. As the student admitted, the 3*n* term was not established through mathematical reasoning but seemed to have been uncovered with some element of luck. In other student responses, the use of the *guess-and-*

check strategy was very obvious from the scribbling on the scripts. The student response in Figure 3.12(b) above shows that Student 69M3 made a few speculations of the rule to see which one exactly fitted the given pattern. Student responses that showed the use of the *guess-and-check* strategy were coded as 31 for generalising strategy used.

Finally, a problematic issue with some of the student justifications lies in the propensity of students to verify the truth of their rules instead of offering an account of how the rules were derived. The justification of Student 18M3 in Figure 3.6 above illuminates this issue. Although the student produced the correct rule, *3 times the size number in addition to 2*, for the *Bricks* task, it remained unclear from the justification how the rule actually emerged. Thus this justification failed to shed light on the generalising strategy used. Like Student 18M3, Student 19M3 in Figure 3.13 below also never described how the rule, $3(n+1)-1$, for the *Bricks* task was developed.


(a) Write down the rule John might have used in terms of the size number.

$3(n+1)-1$

Add 1 to the size number, then multiply by 3. After that ~~minus~~ subtract 1 from it.

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Size 1: $3(n+1)-1$
$3(1+1)-1$
$6-1=5$
5 bricks in size 1

Size 2: $3(2+1)-1$
$\cancel{6}9-1=8$
8 bricks in size 2

Size 3: $3(3+1)-1$
$12-1=11$
11 bricks in size3.

Figure 3.13. Bricks response of Student 19M3

To produce $3(n+1)-1$, one has to view the configurations in Figure n as being made up of three identical rows of $(n+1)$ bricks with one brick missing in the middle row. However, there was no convincing evidence in the test script to confirm that was how Student 19M3 had viewed the configurations. Even though it was tempting to read between the lines when trying to ascertain the generalising strategies used, the researcher sought to resist such a temptation during coding. As a result, the generalising strategies used by both Students 18M3 and 19M3 remained unidentifiable and were, therefore, classified as indeterminate. This type of student responses was assigned Code 41 for the generalising strategy used.

3.4.4 THE CODING SCHEME FOR JUSTIFICATION SCHEMES

The justification schemes in the initial coding scheme were built upon a range of sources, including, the existing classification scheme by Rivera and Becker (2011), the results from the pilot study, and the researcher's observations noted during the scoring of the student responses in the *JuStraGen* test. Rivera and Becker's classification scheme of four types of justification schemes (namely, extension generation, generic case, formula projection and formula appearance match, described in Section 2.6.2) was found to be inadequate for classifying some of the student justifications in the present study. For instance, some students verified the validity of their functional rules using the given cases rather than generating more new cases. So their justifications could not be classified as extension generation. Two other examples of student justifications that did not fall into any of the four categories include an explanation that offered numerical structures of the pattern to demonstrate the inductive derivation of the functional rule, and one that illustrated the steps of obtaining the functional rule through solving equations. Consequently, Rivera and Becker's classification scheme was expanded and reorganised to yield the initial coding scheme.

The initial coding scheme for justifications comprised broadly four categories: (1) justifying recursive rules, (2) justifying functional rules without diagram, (3) justifying functional rules with diagrams, and (4) miscellaneous. The first category comprised numerically-based schemes typically used for justifying a recursive rule: for instance, indicating the differences between consecutive terms listed as a sequence or between consecutive configurations, and organising the terms in a table. The second category included numerical-based schemes for justifying functional rules: for instance, verifying the validity of a correct functional rule, providing numerical structures of configurations, providing the working for obtaining the functional rule by solving equations, and demonstrating evidence of guess and check. In the third category, the correct functional rule was justified by means of a figural-based argument such as offering a generic configuration or a few configurations to illustrate the derivation of the rule. The fourth category included expressing the functional rule in another different mode, providing an irrelevant justification, and providing no justification for the correct functional rule.

Each justification scheme was assigned a two-digit code, following the coding pattern below:

(a) The leftmost digit indicated the category as described above. Digit 1 for the first category, 2 for the second, 3 for the third, and 9 for the last.

(b) The rightmost digit was used to code the different scheme within the category. For instance, providing a few configurations and a generic configuration in the third category were coded as 31 and 32 respectively.

This initial coding scheme was applied to two classes from School M2: one Express and one Normal (Academic). Nearly 70 of all the 515 scripts of the first set of the *JuStraGen* test were coded. It was discovered that some justifications, especially those involving incomplete rules, did not match any of the available codes. Therefore, a new category for justifying incomplete rules was introduced and it comprised two schemes: elaborating an incomplete rule linked to the size number with examples, and elaborating an incomplete rule not linked to the size number with examples. The second version of the coding scheme that thus emerged was tested on another two classes, one Express and one Normal (Academic) from School M2 as well and this time, almost 60 scripts were coded. The coding scheme was found to be more stable, with minor rewording of a couple of descriptors. So the third version of the coding scheme, found in Appendix 9, eventually emerged and was used subsequently to code all the scripts in the *JuStraGen* test. Some *JuStraGen* responses presented in Sections 3.4.1 and 3.4.3 above are illustrated below to give readers a better sense of how the justification schemes were being coded.

The two examples of justifications in Figures 3.12 and 3.13 show the students verifying the validity of their functional rules. In Figure 3.4, the student described how the rule was developed by means of a few configurations. Finally, a manifestation of a student elaborating an incomplete rule not linked to the size number with examples is illustrated in Figure 3.10.

3.5 INTER-RATER RELIABILITY

Inter-rater reliability was established to determine the extent to which different people scoring and coding the responses using the scoring rubric and the coding schemes respectively would get consistent results. Percentage agreement was used to estimate the inter-rater reliability. Based on the proportion of students from these schools in the entire student sample, 36 scripts were selected from each set of *JuStraGen* test: 14 from School M1, 12 from School M2, and 10 from School M3. The 36 (7%) scripts of Set 1, comprising four tasks, were passed to a retired mathematics teacher for scoring and coding, and the other 36 scripts of Set 2 were passed to a secondary school mathematics teacher. A total of 144 tasks were scored and coded for generalising strategies and justification schemes by each teacher. The teachers were not asked to code the types of functional rule and their modality because the coding was straightforward. The researcher explained the scoring rubric and coding schemes to the teachers and trained them to use these before they started to code. Nearly 90% of the scores and over 94% of the codes given by each teacher matched those given by the researcher. Given that scoring student responses using a rubric can be somewhat subjective and open to interpretation, it was hardly surprising there were some disagreements amongst the validators over the scores. However, it was believed that at 90% agreement level amongst them was adequate for the purpose of the present study. Lastly, the high agreement level in coding also indicated the reliability of the coding schemes for generalising strategies and justification schemes.

For data entry, two persons, a friend and the researcher, separately entered the scores and codes into SPSS. A check was then made to compare the two sets of data and a few discrepancies were spotted. After verifying with the original data, the discrepancies were found to be caused by the friend misreading the researcher's hand-written codes on paper. The researcher's entry in SPSS, which was error-free, was used for data analysis.

3.6 DATA ANALYSIS PLAN

This section provides a description of the data analysis used in Study I and Study II.

3.6.1 STUDY I

The first two main research questions of Study I investigated the types of rules formed by students, the modality of the rules, their generalising strategies and justification schemes. A detailed task-by-task analysis was carried out by examining the students' responses for each *JuStraGen* task to identify the four components listed above using the respective coding schemes described in Sections 3.4.2 to 3.4.4. After the inter-rater reliability was established, the frequency of response for each category in each component was then computed using SPSS and separated according to the different pattern formats in each student course.

To illustrate, consider the recursive and functional rules formed in *Bricks* provided in Table 3.18 below. After each student response was matched with the available codes, the frequency of each type of rule was then computed for the four groups of students (i.e., G1 and G2 in Express course; G1 and G2 in Normal (Academic) course) working with the different pattern formats, as well as for the overall total in each course. To gauge the student performance in each task, the success rate was determined by using the following formula:

$$Success\ rate = \frac{number\ of\ students\ giving\ correct\ functional\ rules}{total\ number\ of\ students} \times 100\%$$

As an illustration, there were 170 Express students assigned to the successive pattern format. Since there was a total of 110 students deriving a correct functional rule, the success rate for this group of students was computed as follows:

$$Success\ rate = \frac{110}{170} \times 100\% = 65\%$$

Table 3.18: Frequency of each rule type in Bricks by pattern formats and student courses

Express					Normal (Academic)				
Bricks		Format of pattern display			Bricks		Format of pattern display		
Code	Rule type	S	NS	T	Code	Rule type	S	NS	T
101	$3n + 2$	58	47	105	101	$3n + 2$	13	4	17
103	$2(n + 1) + n$ $(2n + 2) + n$	24	39	63	103	$2(n + 1) + n$ $n + (2n + 2)$	1	2	3
102	$5 + 3(n - 1)$	23	4	27	102	$5 + 3(n - 1)$	2		2
105	$2n + (n + 2)$ $2n + (n + 1) + 1$	4	1	5					
106	$3(n + 1) - 1$		3	3					
107	$4n - (n - 2)$		1	1					
108	$2(n + 2) + n - 2$	1		1					
	Functional	110	95	205		Functional	16	6	22
	%	65	57	61		%	17	7	12
120	Recursive	30	6	36	120	Recursive	53	16	69

S: successive assigned to G1; NS: non-successive assigned to G2; T: total

A part of this study included the comparison of each of the four components by the different student courses, the different formats of pattern display and the different types of functions. Illustrations using the different components and variables are now provided to elaborate how the comparisons were made. To compare how the types of functional rules vary with the different formats of pattern display, the number of different rule types for each *JuStraGen* task was considered. For instance, Table 3.18 shows that Express students assigned to the successive pattern format produced five types of functional rules in *Bricks* and their counterparts working with the non-successive pattern format established six. To analyse how the equivalent forms of functional rules vary with the different types of function, the numbers of different types of equivalent functional rules produced in each linear generalising task and its matching quadratic task were determined, and the outcomes were then represented pictorially using bar charts. For exploring how the modality of the rule compare with the different student courses, the frequencies of the three modes for functional rules: in words (W), in notations (N), and in alphanumeric form (WN), for the Express students working with successive pattern format were compared with those for the Normal (Academic) students working with the same format. A similar comparison between the Express and Normal (Academic) students was done for the non-successive pattern format. As Table 3.19 clearly shows, the N mode was predominant regardless of the student courses.

Table 3.19: Frequency of modality of rule in Bricks by pattern formats and student courses

	Express				Normal (Academic)			
		Bricks				Bricks		
		Rule Modality				Rule Modality		
Format of pattern display	n	W	N	WN	n	W	N	WN
S	170	2	54	9	96	1	14	2
NS	167	4	47	6	82	1	6	

Finally, the variation in generalising strategies by the types of functions was examined by pairing up the linear task with its matching quadratic task. For instance, the generalising strategies used in *Bricks* were compared with those in *Wall Design* because these two were matching tasks. This way of studying how the generalising strategies varied with the types of functions applied to examining variation in justification schemes as well.

The third main research question investigated the effect of the format of pattern display and of the types of functions on students' rule construction. Students' responses for the individual *JuStraGen* generalising tasks were first scored using the analytic scoring rubric described in Section 3.4.1. Next, the scores of the four linear tasks, the four quadratic tasks and all eight tasks were totalled up. The score awarded to each task appeared to be ordinal and there has been an on-going debate on the appropriateness of performing parametric tests on such ordinal-scaled data (Dooley, 2001). This is because a test score of 9 in the *JuStraGen* test is 6 points higher than a score of 3 but it does not mean that a student scoring 9 is three times as knowledgeable as another student scoring 3. However, in this study, the scores of a few tasks were combined as a composite score to measure the students' ability in generalisation. By using a composite score, some social researchers, therefore, believed that the ordinal data could be converted into a form of pseudo-interval data, thereby allowing parametric tests to be applied (Yu, 2002).

To explore the effect of the format of pattern display on students' rule construction, the mean scores and standard deviations for the four linear tasks, the four quadratic tasks, and all eight tasks between successive and non-successive formats were computed for Express and Normal (Academic) students separately. These mean scores were subsequently used to measure the students' performance in those tasks. To determine whether the mean scores for (a) linear tasks, (b) quadratic tasks, and (c) all the tasks between the students working with the two pattern formats differed significantly, three independent *t*-tests were performed for each student course. When the difference was found to be significant, Cohen's *d*, which is a measure of effect size based on means, was reported. An effect size of 0.2 is considered as small, 0.5 as medium and 0.8 as large (Burns, 2000, p. 169).

To investigate the effect of the types of functions on students' rule construction, the mean scores of each matching pair of linear and quadratic tasks were compared for Express and Normal (Academic) students separately. To determine whether the mean score of the linear generalising task differed significantly from that of the quadratic generalising task, four paired *t*-tests were performed for each student course. When the difference was significant, Cohen's *d* was reported. Values of Cohen's *d* on the order of 0.20, 0.50 and 0.80 represent small, medium and large effects sizes respectively.

3.6.2 STUDY II

The fourth main research question explored what the students judged as the most helpful generalising strategy for constructing the functional rule. Their responses in the *QBBS* tasks were examined and then compared with the generalising strategies they used in the *JuStraGen* test. The distributions of best-help generalising strategies between (a) the Express and Normal (Academic) students, and (b) students working with successive and non-successive formats of pattern display were investigated for significant differences. Finally, an investigation was also carried out through interviews to probe the efficacy of the students' choice of best-help generalising strategies on their rule construction.

Each of the four *QBBS* tasks provided four approaches, with each featuring a different generalising strategy, of working out the functional rule. The frequency and percentage for each approach was counted and distinguished by the different pattern formats in each student course to identify which strategy was the students' favourite. Eight χ^2-tests were then performed to determine whether the distributions of strategies in each *QBBS* task between the Express and Normal (Academic) students working with (a) successive pattern format, and (b) non-successive format were significantly different. Another eight χ^2-tests were also conducted to determine whether there were any significant differences in the distributions of strategies in each *QBBS* task between the successive and non-successive format within (a) the Express course, and (b) the Normal (Academic) course. When the difference was found to be significant, the observed and expected frequencies of strategies were computed for further examination.

To check whether the students' choice of best-help strategies in the *QBBS* survey were related to their generalising strategies in the *JuStraGen* test, two cross tabulations of the frequencies of their *QBBS* strategies and *JuStraGen* strategies were generated for each student course: one for the two tasks in *QBBS*1, and another for the other two tasks in *QBBS*2. The efficacy of the students' *QBBS* strategies on their rule construction was established by determining the frequency of students who derived the functional rules correctly during the interviews. All the conversations between the students and the researcher were transcribed to capture rich qualitative data on their generalisations as well as justifications for their *QBBS* strategies. Multiple data sources in the form of video-recordings, audio-recordings and the student writing on the questionnaire were collected. These different kinds of data help to triangulate the observed outcomes of the students' generalisations and offer greater confidence in the validity of the interviews.

3.7 SUMMARY

This chapter has described the research design, subjects, instruments, scoring rubric and coding schemes, data collection, and data analysis involved in Study I and Study II. Study I examined students' strategies and justifications and the influence of two task features through a paper-and-pencil test, and Study II, through a questionnaire, looked at students' beliefs about which strategy would best help them to derive the rule as well as their ability to construct the rule using their choice of strategy.

Study I, involving 515 students, used an independent-measures experimental design to examine whether different formats of pattern display had any effect on students' rule construction and a repeated-measures design to determine whether their rule construction was influenced by the different types of function. Apart from investigating the effect of the two task features, the students' functional rule and its modality, generalising strategies and justification schemes were also examined.

In Study II, a survey study was employed with all 515 students asked to identify their choice of best-help generalising strategy. This was then followed by interviews with 16 students to probe whether they were able to correctly derive the functional rule using their chosen strategy.

Three instruments were devised for this study: the Generalisation Attainment Test (*GAT*) and the Strategies and Justifications in Mathematical Generalisation (*JuStraGen*) Test used in Study I, and Questionnaire of Students' Belief of Best-help Strategies (*QBBS*) in Study II. An analytic scoring rubric was developed to score the students' responses in the *JuStraGen* Test and three coding schemes were developed for coding students' rules and their modality, generalising strategies and justification schemes. The results of Study I and Study II are presented in the next two chapters.

CHAPTER 4 : RESULTS OF STUDY I

Study I aimed to investigate (a) Singapore secondary school students' generalisation and justification of figural patterns with varying formats of pattern display and types of function, and (b) the effect of the format of pattern display and the type of function on the students' generalisations and justifications. The data used in this investigation were drawn from 515 students' scripts of the *JuStraGen* test. The students, comprising 337 from the Express course and 178 from the Normal (Academic) course, were from three different schools: M1, M2 and M3.

This chapter reports the findings about the generalisations and justifications produced by the students in the *JuStraGen* test, as well as the effects of the two task features on these two elements. It begins with an overview of the students' performance in the *JuStraGen* test and of the possible effect of the different task features on their performance in Section 4.1, followed by a more detailed description of the types of rules the students constructed for the general term of a pattern, the modality of their rule and their choice of generalising strategies in Section 4.2. Section 4.3 presents a comprehensive description of the students' justification schemes and Section 4.4 contains a discussion of the possible effect of the format of pattern display and the type of function on the students' generalisations and justifications.

4.1 OVERVIEW OF RESULTS OF STUDY I

This section begins with a synopsis of the students' generalising ability by analysing the success rates of their performance in the *JuStraGen* test, which comprises four linear and another four quadratic tasks. Table 4.1 presents both the number and percentage of students who constructed a correct functional rule for each of the eight generalising tasks by format of pattern display and course. Students assigned to tasks with successive configurations were labelled Group 1 (G1) and those who dealt with non-successive configurations were labelled Group 2 (G2).

Table 4.1: Frequency and percentages of successful students in JuStraGen test by format of pattern display and course

Course	Format of pattern display	Linear tasks				Quadratic tasks			
		Bricks	**Birthday Party Decorations**	**Towers**	**High Chairs**	**Oh Deer!**	**Tulips**	**Christmas Party Decorations**	**Wall Design**
Express	S (n = 170)	110 (65%)	117 (69%)	115 (68%)	105 (62%)	93 (55%)	88 (52%)	90 (53%)	88 (52%)
	NS (n = 167)	95 (57%)	106 (63%)	117 (70%)	116 (69%)	88 (53%)	93 (56%)	102 (61%)	83 (50%)
Normal (Academic)	S (n = 96)	16 (17%)	17 (18%)	17 (18%)	16 (17%)	8 (8%)	8 (8%)	11 (11%)	7 (7%)
	NS (n = 82)	6 (7%)	8 (10%)	6 (7%)	7 (9%)	5 (6%)	2 (2%)	6 (7%)	3 (4%)

S: successive, NS: non-successive

The table reveals that students in both courses were generally more successful in linear generalising tasks than in quadratic generalising tasks. In the Express course, the success rates of G1 students in linear tasks spanned from 62% to 69%, but the percentages dipped to between 52% and 55% for quadratic tasks. The success rates of G2 students, ranging from 57% to 70% for linear tasks and from 50% to 61% for quadratic tasks, were almost comparable to those of G1 students. A similar trend was observed in the Normal (Academic) course, albeit much lower success rates. The percentages of successful G1 students were about 18% for linear tasks and varied between 7% and 11% for quadratic tasks. In contrast, the G2 students fared less well, with success rates ranging from 7% to 10% for linear tasks and a low of 2% to 7% for quadratic tasks. This shows that there was a wide variation in performance between the Express and Normal (Academic) students.

Comparing the student performance in both groups, G1 students in the Normal (Academic) course outperformed their counterparts in G2 in *all* the eight tasks. On the other hand, Express students in G1 had higher success rates than their G2 counterparts in only four tasks: two linear (*Bricks* and *Birthday Party Decorations*), and two quadratic (*Oh Deer!* and *Wall Design*). In the remaining four tasks, G2 Express students did better than their counterparts in G1.

These findings indicate that a sizeable number of Express students answered linear generalising tasks well; however, they seemed to flounder when dealing with quadratic tasks. To them, the linear tasks appeared simpler to do than the quadratic tasks. Yet these seemingly easier linear tasks provided a stern challenge to a vast majority of the Normal (Academic) students. It was therefore not a surprise to find them facing an even greater challenge with the quadratic generalising tasks. Further, the G2 students from this course performed less well in all *JuStraGen* tasks in comparison with their G1 counterparts.

All the generalising tasks in the *JuStraGen* test required students to express the rule for directly calculating any output value specifically in terms of the size number. So a functional, and not recursive, rule was expected in the answer. Table 4.2 and Table 4.3 below present both the percentages of students producing the different types of rules and the number of different equivalent functional rules constructed correctly by the students in the Express and Normal (Academic) courses respectively. The rules were broadly categorised as *functional* (F), *recursive* (R) or *others* (O).

Table 4.2 shows that the rules produced by the Express students were predominantly functional in nature. Of the Express students in G1, the percentages of correct functional rules spanned from 62% in *High Chairs* to 69% in *Birthday Party Decorations* whilst those for quadratic generalising tasks varied between 52% in *Wall Design* to 55% in *Oh Deer!*. The outcomes for the Express students in G2 were similar, with success rates between 57% in *Bricks* and 70% in *Towers* for linear tasks and between 50% in *Wall Design* and 61% in *Christmas Party Decorations* for quadratic tasks. On the other hand, less than a fifth of the Express students produced a correct recursive rule.

The Express students also produced a wider diversity of equivalent functional rules for each generalising task in the *JuStragen* test. The modal number of functional rules generated was 8, with the least in *Birthday Party Decorations* (6 types) and the most in *High Chairs* (13 types).

Table 4.2: Percentage of Express students by rule type and number of equivalent functional rules produce

Format of pattern display	n	Bricks			Birthday Party Decorations			Towers			High Chairs		
		Rule Type			Rule Type			Rule Type			Rule Type		
		F	R	O	F	R	O	F	R	O	F	R	O
S	170	65	17	18	69	17	14	68	16	16	62	17	21
NS	167	57	4	39	63	12	25	70	9	21	69	10	21
	EFR	7			6			12			13		

		Oh Deer!			Tulips			Christmas Party Decorations			Wall Design		
		Rule Type			Rule Type			Rule Type			Rule Type		
	n	F	R	O	F	R	O	F	R	O	F	R	O
S	170	55	9	36	52	11	37	53	9	38	52	8	40
NS	167	53	3	44	56	5	39	61	3	36	50	5	45
	EFR	8			10			8			7		

S: successive, NS: non-successive, F: function, R: recursive, O: others, EFR: number of equivalent functional rules produced

The results of the Normal (Academic) students demonstrated a great contrast compared to those of their Express counterparts. The percentages of correct functional rules were no more than 20% in all the eight generalising tasks. Looking at the percentages of correct functional and recursive rules in Table 4.3, there were over twice as many recursive rules as functional rules in all the generalising tasks, except for *Christmas Party Decorations*. The difference in the percentages of recursive and functional rules were most noticeable in the linear tasks, with nearly half the G1 students giving a recursive rule. It is evidently clear from these findings that many of the Normal (Academic) students who recognised the pattern did not appreciate the task requirement and their rules were often recursive in nature. As very few of them were successful in the tasks, the number of different equivalent functional rules was also not as many and varied as that derived by the Express students.

The least number of equivalent functional rules was two in *Birthday Party Decorations*, *Tulips* and *Wall Design*; and the maximum was six in *High Chairs*.

Table 4.3: Percentage of Normal (Academic) students by rule type and number of equivalent functional rules produced

Format of pattern display	n	Bricks			Birthday Party Decorations			Towers			High Chairs		
		Rule Type			Rule Type			Rule Type			Rule Type		
		F	R	O	F	R	O	F	R	O	F	R	O
S	96	17	55	28	18	52	30	18	46	36	17	45	38
NS	82	7	20	73	10	28	62	7	43	50	9	46	45
	EFR	3			2			5			6		

		Oh Deer!			Tulips			Christmas Party Decorations			Wall Design		
		Rule Type			Rule Type			Rule Type			Rule Type		
	n	F	R	O	F	R	O	F	R	O	F	R	O
S	96	8	15	77	8	19	73	11	8	81	7	18	75
NS	82	6	12	82	2	9	89	7	2	91	4	5	91
	EFR	4			2			4			2		

S: successive, NS: non-successive, F: function, R: recursive, O: others, EFR: number of equivalent functional rules produced

Both Table 4.4 and Table 4.5 display a breakdown of the different modalities of the functional rules in each *JuStraGen* task by the format of pattern display. The first table presents the outcomes produced by the Express students and the second table displays those by the Normal (Academic) students. The functional rules were found to be articulated in one of the following three modes of representation: *in words* (W), *in notations* (N), and *in alphanumeric form* (WN).

Table 4.4: Percentage of successful Express students by types of rule modality

Format of pattern display	n	Bricks			Birthday Party Decorations			Towers			High Chairs		
		Rule Modality			Rule Modality			Rule Modality			Rule Modality		
		W	N	WN	W	N	WN	W	N	WN	W	N	WN
S	170	2	54	9	3	58	8	2	58	8	1	53	8
NS	167	4	47	6	3	53	7	2	61	7	1	60	8
		Oh Deer!			Tulips			Christmas Party Decorations			Wall Design		
	n	W	N	WN	W	N	WN	W	N	WN	W	N	WN
S	170	3	47	5	4	44	4	4	44	5	2	44	6
NS	167	2	47	4	4	46	6	2	54	5	2	42	6

S: successive, NS: non-successive, W: written in words, N: written in notations, WN: written in alphanumeric form

Table 4.5: Percentage of successful Normal (Academic) students by types of rule modality

Format of pattern display	n	Bricks			Birthday Party Decorations			Towers			High Chairs		
		Rule Modality			Rule Modality			Rule Modality			Rule Modality		
		W	N	WN	W	N	WN	W	N	WN	W	N	WN
S	96	1	14	2	1	14	3	2	14	2	2	10	5
NS	82	1	6			9	1	1	5	1		6	3
		Oh Deer!			Tulips			Christmas Party Decorations			Wall Design		
	n	W	N	WN	W	N	WN	W	N	WN	W	N	WN
S	96	1	6	1		8				5	1	5	1
NS	82		5	1		1	1		6	1		3	1

S: successive, NS: non-successive, W: written in words, N: written in notations, WN: written in alphanumeric form

Of the three modes, expressing the rule *in notations* (N) was the most prevalent category even though the *JuStraGen* generalising tasks did not specify any particular mode of representation for rule construction. As the two tables show, over 40% of the Express students and as many as 14% of the Normal (Academic) students expressed their functional rules accurately in notations. The use of the other two modes in writing the correct functional rule, on the other hand, was comparatively infrequent. The finding suggests that most of the successful students were capable of using letters to represent numerical values. This is one crucial skill in the learning of algebra that Kaput (2008) had hoped generalising tasks will help students to develop.

Turning now to the analysis of student use of generalising strategies for establishing the functional rule correctly, several strategies were disclosed and these were classified into four broad categories: *numerical* (N), *figural* (F), *guess-and-check* (G) and *indeterminate* (I). Numerical strategies included constructing a rule by expressing each subsequent term in a pattern in terms of the immediate term preceding it and by comparing the terms in a pattern with corresponding terms of another pattern whose rule is already known. Examples of figural strategies included breaking up the original configuration into smaller non-overlapping parts, as well as rearranging one or more parts of the original configuration to form something more familiar. A *guess-and-check* strategy was engaged when different algebraic expressions were tested until finding one that fitted those few terms under consideration. Student responses that did not illuminate clearly how a correct rule was derived were classified as having used an *indeterminate* strategy. The percentages of successful Express and Normal (Academic) students in each category of the generalising strategies are summarised in Table 4.6 and Table 4.7 respectively.

Table 4.6 discloses clearly that there was widespread use of figural strategies amongst the Express students. In *Birthday Party Decorations*, *Towers*, *High Chairs, Christmas Party Decorations* and *Oh Deer*!, at least 33% of the G1 students used them to work out the rules and the percentages of G2 students surged to nearly 50% in the first four tasks. In *Bricks,* the *figural* strategies were not only the top favourite of students in G2, but also in G1, alongside *numerical* and *indeterminate* strategies. However, in *Tulips,* the percentage of *guess-and-check* was highest for both G1 and G2, with the *figural* and *indeterminate* strategies equally prevalent in the latter group. Finally, the Express students showed preference for *numerical* strategies over *figural* strategies in *Wall Design*.

As Table 4.6 manifests clearly, there were more G1 than G2 students employing the numerical approach. For instance, the percentages of G1 and G2 students in *Bricks* were 19% and 3% respectively whereas the values were 27% and 18% respectively in *Wall Design*. Therefore, this finding reflects the Express students' propensity to use the numerical approach when the configurations were presented sequentially.

Another noticeable outcome of the strategy analysis was the number of students' generalising strategies that were classified as *indeterminate* in generalising tasks such as *Bricks*, *Birthday Party Decorations*, *Towers* and *Tulips*. The percentages in both student groups were all over 10% and these were not small values. This is why the results deserved some attention. This evidence suggests that some Express students were unfamiliar with the expectations of the justification that were asked of them. As for using *guess-and-check* to obtain the rules, the percentages of G1 and G2 students were small at below 10% in all the tasks, with the only exception of *Tulips*.

Unlike the clear preference of successful Express students for a figural approach, the successful Normal (Academic) students' choices of generalising strategies were rather ambivalent across all the eight *JuStraGen* tasks, as Table 4.7 indicates. This situation happened because the low success rates were spread thinly amongst the four different categories of generalising strategies. Whilst there appeared to be a predominant use of a figural approach in *Towers* (7% in G1, 4% in G2), *High Chairs* (11% in G1, 6% in G2), *Oh Deer!* (4% in G2) and *Christmas Party Decorations* (8% in G1, 4% in G2) and the numerical approach only in *Wall Design* (4% in G1, 3% in G2), the students' strategies were largely indeterminate in the remaining three tasks of *Bricks* (11% in G1, 4% in G2), *Birthday Party Decorations* (10% in G1, 6% in G2) and *Tulips* (5% in G1, 2% in G2). Interestingly, the numerical approach was spotted in G1 in nearly all the tasks, with the exception of *Oh Deer!* and *Christmas Party Decorations*. None of the G2 students used it in any of the tasks except for *Wall Design* only. This finding corroborates an earlier result found in the strategy analysis of the Express students that figural patterns displaying the configurations successively tend to invoke the use of the numerical approach.

Table 4.6: Percentage of successful Express students by types of generalising strategies

Format of pattern display	n	Bricks				Birthday Party Decorations				Towers*				High Chairs			
		Generalising Strategies				Generalising Strategies				Generalising Strategies				Generalising Strategies			
		N	F	G	I	N	F	G	I	N	F	G	I	N	F	G	I
S	170	19	20	5	21	16	34	3	16	12	38	7	10	14	35	5	8
NS	167	3	34	4	16	4	45	2	12	4	45	5	16	4	49	2	14

	n	Oh Deer!				Tulips				Christmas Party Decorations				Wall Design			
		Generalising Strategies				Generalising Strategies				Generalising Strategies				Generalising Strategies			
		N	F	G	I	N	F	G	I	N	F	G	I	N	F	G	I
S	170	4	33	9	9	5	13	20	14	1	39	5	8	27	16	5	4
NS	167	2	35	7	9	3	19	17	17		48	1	12	18	14	7	11

S: successive, NS: non-successive, N: numerical, F: figural, G: guess and check, I: indetermindate
* exclude one student (1%) who obtained a correct rule using an incorrect strategy

Table 4.7: Percentage of successful Normal (Academic) students by types of generalising strategies

Format of pattern display	n	Bricks				Birthday Party Decorations				Towers				High Chairs			
		Generalising Strategies				Generalising Strategies				Generalising Strategies				Generalising Strategies			
		N	F	G	I	N	F	G	I	N	F	G	I	N	F	G	I
S	96	2	3	1	11	2	5	1	10	2	7	4	4	1	11	3	2
NS	82		2	1	4		3	1	6		4	1	2		6	1	1

	n	Oh Deer!				Tulips				Christmas Party Decorations				Wall Design			
		Generalising Strategies				Generalising Strategies				Generalising Strategies				Generalising Strategies			
		N	F	G	I	N	F	G	I	N	F	G	I	N	F	G	I
S	96		3	4	1	1	1	1	5		8	3		4	3		
NS	82		4		2				2		4		4	3	1		

S: successive, NS: non-successive, N: numerical, F: figural, G: guess and check, I: indetermindate

The analysis of the justification schemes used by students to describe how they formulated their functional rules shows that the schemes fell largely into three categories, viz, *justifying without diagrams* (F), *justifying with diagrams* (FD) and *miscellaneous* (M). Examples of justification schemes without using diagrams comprise, for instance, validating a functional rule by substituting some values into it, providing a few numerical structures of the pattern that describing in words the steps taken to derive a functional rule by means of comparing the pattern in question with another known sequence. Manifestations of justification schemes supported by diagrams occur when students provided a generic or a few configurations to illustrate how the pattern structure was visualised and linked to the functional rule. Aside from these justification schemes, producing simply configurations not provided in the *JuStraGen* tasks without any accompanying explanation and repeating the rule in a different mode of representation, such as expressing the algebraic expression for the general term of the pattern in words and vice versa, were two examples of *miscellaneous* justification schemes observed in some student justifications. Justification schemes meant for explaining a recursive rule (R) were used only in the first set of the *JuStraGen* test. Table 4.8 and Table 4.9 tabulate the percentages of successful students who engaged in the different types of justification schemes in the Express and Normal (Academic) courses respectively.

Table 4.8: Percentage of successful Express students by types of justification schemes

Format of pattern display		Bricks				Birthday Party Decorations				Towers				High Chairs			
		Justification Schemes				Justification Schemes				Justification Schemes				Justification Schemes			
	n	R	F	FD	M	R	F	FD	M	R	F	FD	M	R	F	FD	M
S	170	1	38	18	8	1	36	26	5		25	33	9		24	28	10
NS	167	2	26	24	4	1	22	37	4		20	38	13		23	40	6

		Oh Deer!				Tulips				Christmas Party Decorations				Wall Design			
		Justification Schemes				Justification Schemes				Justification Schemes				Justification Schemes			
	n	R	F	FD	M	R	F	FD	M	R	F	FD	M	R	F	FD	M
S	170		25	26	3		29	15	7		18	28	7		28	19	5
NS	167	1	20	28	5	1	30	20	5		15	37	10		22	19	10

S: successive, NS: non-successive, R: justifying recursive rule, F: justifying functional rule without diagram, FD: justifying functional rule with diagrams, M: miscellaneous

Table 4.8 indicates that a considerable proportion of Express students justified their functional rules using F or FD. These two types of justification schemes were equally popular amongst the G1 students, with F predominantly used in *Bricks* (38%), *Birthday Party Decorations* (36%), *Tulips* (29%) and *Wall Design* (28%), and FD in the remaining four tasks: *Towers* (38%), *High Chairs* (40%), *Oh Deer!* (28%) and *Christmas Party Decorations* (37%). On the other hand, G2 students appeared to favour FD over F, judging from its prevalent use in five *JuStraGen* tasks – namely, *Birthday Party Decorations* (37%), *Towers* (38%), *High Chairs* (40%), *Oh Deer!* (28%) and *Christmas Party Decorations* (37%) – as well as its high occurrence in the rest of the tasks. Inferring from the results, students in G2 might have found it easier and more straightforward to relate their rules to the given configurations in their justifications. In line with these findings, the merit of non-successive configurations in promoting the detection and recognition of pattern structure was once again validated. Another outcome emerging from the analysis of justification schemes in both groups of students is the percentages of M which hovered around 10% in certain *JuStraGen* tasks such as *Towers* (13% in G2) and *High Chairs* (10% in G1). The values might be considerably low in comparison with the percentages of F and FD but, taking into account the students' prior attainment, they were deemed sizeable enough to warrant mentioning. Finally, a couple of students (1 – 2%) justified their rules in the first set of *JuStrGen* test using R and none did so in the second set of test. Taken together, these latter findings point to a lack of understanding of the justification requirements on the part of the Express students because their responses did not address how they obtained their rules.

Table 4.9: Percentage of successful Normal (Academic) students by types of justification schemes

Format of pattern display	n	Bricks				Birthday Party Decorations				Towers				High Chairs			
		Justification Schemes				Justification Schemes				Justification Schemes				Justification Schemes			
		R	F	FD	M	R	F	FD	M	R	F	FD	M	R	F	FD	M
S	96	1	10	1	4	1	8	3	5		8	5	4		7	6	3
NS	82		2	2	2		2	2	5			2	5		1	6	1

	n	Oh Deer!				Tulips				Christmas Party Decorations				Wall Design			
		Justification Schemes				Justification Schemes				Justification Schemes				Justification Schemes			
		R	F	FD	M	R	F	FD	M	R	F	FD	M	R	F	FD	M
S	96		1	4	3		5	1	2		2	7	2		4	2	1
NS	82		1	4	1		1		1			5	2			2	1

S: successive, NS: non-successive, R: justifying recursive rule, F: justifying functional rule without diagram, FD: justifying functional rule with diagrams, M: miscellaneous

As for the Normal (Academic) students, the analysis of their justification schemes in Table 4.9 reveals fairly similar results to those of the Express students. Comparing the percentages of G2 students in F and FD, FD had higher values in the following five *JuStraGen* tasks: *Towers* (2%), *High Chairs* (6%), *Oh Deer!* (4%), *Christmas Party Decorations* (5%) and *Wall Design* (2%). As well, the percentages of students applying M were somewhat moderate relative to the other categories in certain particular tasks such as *Bricks*, *Birthday Party Decorations* and *Towers*. Another similarity between the Express and Normal (Academic) students is that the number of students employing R in their justifications remained very small. This scheme was used only by G1 students in *Bricks* (1%) and *Birthday Party Decorations* (1%). What is dissimilar from the findings of the Express students in G1 is that F was a clear favourite amongst the Normal (Academic) students in G1, drawing evidence from the higher percentages in F than in FD in the following six tasks: *Bricks* (10%), *Birthday Party Decorations* (8%), *Towers* (8%), *High Chairs* (7%), *Tulips* (5%) and *Wall Design* (4%).

The present study had shed light on the potential effect of two task features on students' generalisations. There was empirical evidence to show that the format of pattern display could influence students' generalisations. The Normal (Academic) students had found generalising tasks with non-successive configurations significantly harder to do than those with successive configurations. In contrast, the format of pattern display did not appear to have any significant effect on the more able students in the Express course. As for the second task feature of the type of functions, quadratic generalising tasks were found to be significantly more challenging than linear generalising tasks for both Express and Normal (Academic) students.

Following this brief overview on both the students' performance in the *JuStraGen* test and the effect of the format of pattern display and the type of functions on their generalisation and justification, it is worthwhile examining the specific performance of students on each of these aspects. Subsequent sections provide a more detailed discussion of their performance and the possible effect of both task features as evident in the test.

4.2 STUDENTS' PERFORMANCE IN JUSTRAGEN TEST

This section focusses on the first two main research questions, beginning with an examination of how students established their rules, followed by an investigation of their justifications of the rules. The first research question covers the following three aspects: the types of rules that the students constructed for finding the general term of a pattern, the modalities of their rules and the types of generalising strategies they employed to derive the rules. The second research question explores the types of justification schemes that they used to explain their rules.

4.2.1 FORMULATION OF RULES

The first main research question is stated below:

1. *How do Singapore secondary school students establish the rule that defines a figural pattern?*

4.2.1.1 Types of rules formulated

The aim of this section is to identify the different types of rules constructed by students when they dealt with linear and quadratic generalising tasks. All the rules for each generalising task were analysed comprehensively and several types of equivalent functional rules were revealed. After further examination, the various types which had similar structure were collapsed into the same category. Following the establishment of the distinct types of functional rules for each task, an attempt was then carried out to determine how the types of functional rules compared between the Express and Normal (Academic) students, between successive and non-successive pattern formats, as well as between linear and quadratic patterns. Results from these comparisons are discussed below.

1.1.1 What are the different forms of rules that the students formulate for a figural pattern?

As explained previously in Section 3.3.2, the initial written expression of the functional rule, albeit simplified to another form subsequently, was coded. So when the expression, $5 + 3(n - 1)$, was first constructed and then simplified to its closed form, $3n + 2$, the former expression was coded. In other words, simplifying a functional rule to its closed form was not required. As a result, different equivalent functional rules were produced for each generalising task. The various linear and quadratic functional rules produced by the Express students, arranged in decreasing order of occurrence, are listed in Table 4.10 and Table 4.11 respectively below.

Table 4.10 indicates that the Express students produced seven categories of different but equivalent expressions of linear functional rules in *Bricks*, six in *Birthday Party Decorations*, 12 in *Towers* and 13 in *High Chairs*. Table 4.11 illustrates that the number of categories of functional rules in each quadratic generalising task was just as many, with eight each in *Oh Deer!* and *Christmas Party Decorations*, 10 in *Tulips* and seven in *Wall Design*.

Table 4.10: Distribution of successful Express students by rule type and format of pattern display for linear generalising tasks ($n_S = 170$, $n_{NS} = 167$, $n_T = 337$)

Bricks		Format of pattern display			Birthday Party Decorations		Format of pattern display			Towers		Format of pattern display			High Chairs		Format of pattern display		
Code	Rule type	S	NS	T	Code	Rule type	S	NS	T	Code	Rule type	S	NS	T	Code	Rule type	S	NS	T
101	$3n+2$	58	47	105	301	$3n+2$	88	94	182	603	$2(n+1)+2n$	61	40	101	801	$3n+5$	46	61	107
103	$2(n+1)+n$ $(2n+2)+n$	24	39	63	302	$5+3(n-1)$	18	5	23	601	$4n+2$	32	46	78	804	$3(n+1)+2$	16	23	39
102	$5+3(n-1)$	23	4	27	303	$2(n+1)+n$	7	3	10	602	$6+4(n-1)$	6	7	13	803	$2(n+1)+(n+3)$	19	14	33
105	$2n+(n+2)$ $2n+(n+1)+1$	4	1	5	304	$3(n+2)-4$	1	2	3	609	$2n+2n+2$ $2(2n)+2$	3	10	13	805	$3(n+2)-1$	4	11	15
106	$3(n+1)-1$		3	3	306	$2n+(n+2)$	2	1	3	604	$2n+(n+2)+n$ $3n+(n+2)$	3	6	9	802	$8+3(n-1)$	7	5	12
107	$4n-(n-2)$		1	1	305	$3(n+1)-1$	1	1	2	607	$10+4(n-2)$	6		6	807	$11+3(n-2)$	5		5
108	$2(n+2)+n-2$	1		1						606	$(n+1)(n+2)-n(n-1)$	1	2	3	809	$6+(3n-1)$	3		3
										608	$2(2n+1)$	1	2	3	811	$2(n+2)+(n+1)$	1	1	2
										610	$2(n+2)+2(n-1)$		2	2	806	$3(n+3)-4$		1	1
										612	$5n-(n-2)$	2		2	808	$2(n+3)+(n-1)$	1		1
										611	$(n+2)^2-n^2-2$		1	1	810	$5(n+1)-2n$	1		1
										616	$4\left(n+\frac{1}{2}\right)$		1	1	812	$(3n-1)+6$	1		1

															813	$4n + 4 - (n - 1)$	1		1
	Functional	110	95	205		Functional	117	106	223		Functional	115	117	232		Functional	105	116	221
	%	65	57	61		%	69	63	66		%	68	70	69		%	62	69	66
120	Recursive	30	6	36	320	Recursive	28	19	47	620	Recursive	27	15	42	820	Recursive	28	16	44
960	$n + 3$	4	4	8	960	$n + 3$	4	8	12	960	$n + 4$	2	7	9	960	$n + 3$	3	5	8
950	Workable rule not in terms of n		5	5															

S: successive, NS: non-successive, T: total

Table 4.11: Distribution of successful Express students by rule type and format of pattern display for quadratic generalising tasks ($n_S = 170$, $n_{NS} = 167$, $n_T = 337$)

Oh Deer!		Format of pattern display			Tulips		Format of pattern display			Christmas Party Decorations		Format of pattern display			Wall Design		Format of pattern display		
Code	Rule type	S	NS	T	Code	Rule type	S	NS	T	Code	Rule type	S	NS	T	Code	Rule type	S	NS	T
203	$n(n+1)+2(n+1)$ $n(n+1)+2n+2$	49	42	91	402	$n(n+2)$	53	50	103	502	$n(n+2)+2$ $n(3+n-1)+2$	50	62	112	702	$(n+1)^2$	63	42	105
202	$(n+1)(n+2)$	25	16	41	401	n^2+2n	19	25	44	503	$n^2+2(n+1)$	25	14	39	701	n^2+2n+1 $n^2+(2n+1)$	13	17	30
204	$n^2+2n+(n+2)$ $n(n+2)+(n+2)$	6	16	22	403	$n+n(n+1)$ $n+(n^2+n)$ $n+\frac{2n(n+1)}{2}$	7	9	16	501	n^2+2n+2	6	20	26	703	$n(n+2)+1$	5	14	19
201	n^2+3n+2	4	6	10	405	$3n+n(n-1)$	3	3	6	506	$(n+1)^2+1$	5	1	6	704	$n(n+1)+(n+1)$	5	6	11
205	$n^2+n+2(n+1)$	2	5	7	406	$(n+1)^2-1$	5	1	6	504	$(n+1)(n+2)-n$	1	4	5	707	$n(n-1)+3n+1$	2	2	4
208	$(n+1)^2+(n+1)$	3	2	5	408	$n(2n+1)-n(n-1)$		2	2	505	$n(n+1)+n+2$	1	1	2	705	$4n+(n-1)^2$		1	1
206	$n(n+3)+2$	3	1	4	407	$(2n+1)(n+1)$ $-n(n+1)-1$		1	1	507	$n(n-1)+3n+2$	1		1	706	$(2n+1)(n+1)$ $-n(n+1)$		1	1
209	$n^2+8+3(n-2)$	1		1	409	$8+(n+4)(n-2)$	1		1	508	$(n+1)^2-2(n+1)$	1		1					
					410	$(n^2-1)+(n+1)+n$		1	1										

					404	$n + 2[n + (n-1) + (n-2) + \cdots + 3 + 2 + 1]$		1	1										
	Functional	93	88	181		Functional	88	93	181		Functional	90	102	192		Functional	88	83	171
	%	55	53	54		%	52	56	54		%	53	61	57		%	52	50	51
220	Recursive	17	4	21	420	Recursive	16	6	22	520	Recursive	12	1	13	720	Recursive	9	6	15
221	First difference identified only		2	2	421	First difference identified only	3	3	6	521	First difference identified only	3	4	7	721	First difference identified only	5	1	6
960	$n + (multiples\ of\ 2)$	1	1	2	960	$n + (odd\ numbers)$	1		1	960	$n + (odd\ numbers)$	1		1	960	$n + (odd\ numbers)$		2	2
					950	Workable rule not in terms of n	2		2										

S: successive, NS: non-successive, T: total

Table 4.12 and Table 4.13 below provide respectively the various linear and quadratic functional rules produced by the Normal (Academic) students, arranged in decreasing order of occurrence. Their rules formed a subset of those generated by their Express counterparts.

As can be seen in Table 4.12, the Normal (Academic) students produced three categories of equivalent expressions of linear functional rules in *Bricks*, two in *Birthday Party Decorations*, and five each in *Towers* and *High Chairs*. Table 4.13 illustrates that the number of categories of functional rules in each quadratic generalising task was about the same, with four each in *Oh Deer!* and *Christmas Party Decorations*, and two each in *Tulips* and *Wall Design*.

Table 4.12: Distribution of successful Normal (Academic) students by rule type and format of pattern display for linear generalising tasks ($n_S = 96$, $n_{NS} = 82$, $n_T = 178$)

Bricks		Format of pattern display			Birthday Party Decorations		Format of pattern display			Towers		Format of pattern display			High Chairs		Format of pattern display		
Code	Rule type	S	NS	T	Code	Rule type	S	NS	T	Code	Rule type	S	NS	T	Code	Rule type	S	NS	T
101	$3n+2$	13	4	17	301	$3n+2$	15	8	23	601	$4n+2$	11	3	14	801	$3n+5$	7	4	11
103	$2(n+1)+n$ $n+(2n+2)$	1	2	3	302	$5+3(n-1)$	2		2	603	$2(n+1)+2n$	3	2	5	804	$3(n+1)+2$	5	2	7
102	$5+3(n-1)$	2		2						608	$2(2n+1)$	2		2	803	$2(n+1)+(n+3)$	1	1	2
										607	$10+4(n-2)$	1		1	802	$8+3(n-1)$	1		1
										609	$2n+2n+2$ $2(2n)+2$		1	1	805	$3(n+2)-1$	1		1
															807	$11+3(n-2)$	1		1
	Functional	16	6	22		Functional	17	8	25		Functional	17	6	23		Functional	16	7	23
	%	17	7	12		%	18	10	14		%	18	7	13		%	17	9	13
120	Recursive	53	16	69	320	Recursive	50	23	73	620	Recursive	44	35	79	820	Recursive	44	38	82
960	$n+3$	10	6	16	960	$n+3$	9	9	18	960	$n+4$	5	8	13	960	$n+3$	8	5	13

S: successive, NS: non-successive, T: total

Table 4.13: Distribution of successful Normal (Academic) students by rule type and format of pattern display for quadratic generalising tasks ($n_S = 96$, $n_{NS} = 82$, $n_T = 178$)

Oh Deer!		Format of pattern display			Tulips		Format of pattern display			Christmas Party Decorations		Format of pattern display			Wall Design		Format of pattern display		
Code	Rule type	S	NS	T	Code	Rule type	S	NS	T	Code	Rule type	S	NS	T	Code	Rule type	S	NS	T
203	$n(n+1)+2(n+1)$ $n(n+1)+2n+2$	3	5	8	402	$n(n+2)$	7	2	9	502	$n(n+2)+2$ $n(3+n-1)+2$	5	1	6	702	$(n+1)^2$	7	2	9
202	$(n+1)(n+2)$	3		3	406	$(n+1)^2-1$	1		1	503	$n^2+2(n+1)$	3	1	4	701	n^2+2n+1 $n^2+(2n+1)$		1	1
206	$n(n+3)+2$	1		1						506	$(n+1)^2+1$	2	2	4					
208	$(n+1)^2+(n+1)$	1		1						501	n^2+2n+2	1	2	3					
	Functional	8	5	13		Functional	8	2	10		Functional	11	6	17		Functional	7	3	10
	%	8	6	7		%	8	2	6		%	11	7	10		%	7	4	6
220	Recursive	9	7	16	420	Recursive	12	6	18	520	Recursive	2	1	3	720	Recursive	10	3	13
221	First difference identified only	5	3	8	421	First difference identified only	6	1	7	521	First difference identified only	6	1	7	721	First difference identified only	7	1	8
960	$n+(multiples\ of\ 2)$	2	2	4	960	$n+(odd\ numbers)$	2	2	4	960	$n+(odd\ numbers)$	1		1	960	$n+(odd\ numbers)$	1		1

S: successive, NS: non-successive, T: total

Consider *Birthday Party Decorations* in Table 4.10 with the following six equivalent rules: $3n + 2$, $5 + 3(n-1)$, $2(n+1) + n$, $3(n+2) - 4$, $2n + (n+2)$ and $3(n+1) - 1$. The rules displayed variation in the mathematical operations used to join different terms together, involving both addition and subtraction. The four rules, $3n + 2$, $5 + 3(n-1)$, $2(n+1) + n$, and $2n + (n+2)$ illustrated the sum of two terms whereas the remaining two rules, $3(n+2) - 4$ and $3(n+1) - 1$, exemplified the difference of two terms. This observation was noticed in other *JuStraGen* tasks as well, where further examples of rules involving subtraction included $(n+2)^2 - n^2 - 2$ in *Towers*, $5(n+1) - 2n$ in *High Chairs*, $n(2n+1) - n(n-1)$ in *Tulips*, and $(n+1)^2 - 2(n+1)$ in *Christmas Party Decorations*. Consistent with the finding of Rivera and Becker (2007), the frequencies of functional rules involving subtraction were normally very low, with not more than six cases in each rule.

A few functional rules in certain *JuStraGen* tasks were particularly worth mentioning because of the thinking and reasoning that students engaged in when they formulated the rules. One prime example was the rule $2n + (n+2)$ (Code 306) constructed by Student 7M1 for *Birthday Party Decorations*, as shown in Figure 4.1.

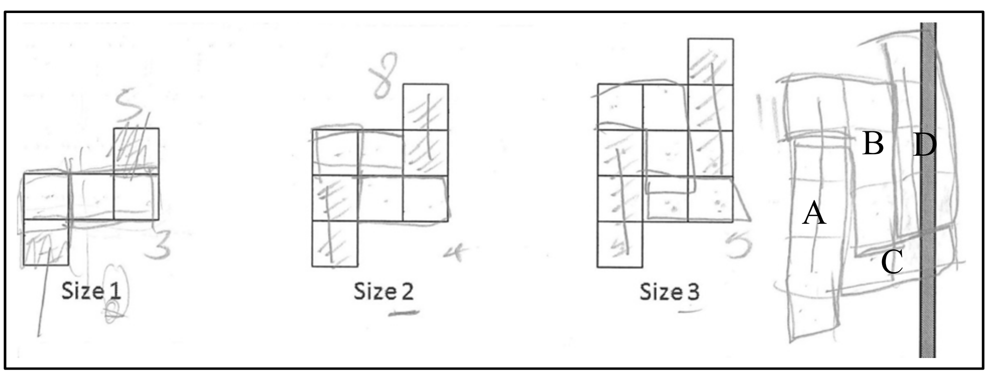


Figure 4.1. Visual representation of $2n + (n+2)$

The rightmost generic configuration, drawn by the student and labelled by the researcher, portrayed clearly how the student discerned and reasoned about the pattern structure in an intriguing manner. First, two identical rectangles, A and D, each comprising *n* square cards, were cut out from the first and third columns. The remaining portion of the configuration was further divided into two parts, B and C. B was a 7-shaped figure, containing *n* square cards and C was a two-square horizontal rectangle. Adding up the $2n$ square cards in A and D and the $(n+2)$ square cards in B and C produced the rule $2n + (n + 2)$. This way of visualising the pattern structure is somewhat unconventional, hence it is worth highlighting.

Figure 4.2 below shows another example regarding the rule $(n + 1)(n + 2) - (n - 1)n$ (Code 606) constructed by Student 37M1 for *Towers*. By rearranging the original configuration into a rectangle with $(n + 1)(n + 2)$ tiles, the student knew that this expression was not the correct formula for finding the actual number of tiles in any configuration. The rule needed adjustment, which was to remove $(n - 1)n$ tiles from the rectangle. Hence, the rule for finding the number of tiles in any configuration was $(n + 1)(n + 2) - (n - 1)n$. This rule takes on an interesting form, which might be easily mistaken for a quadratic function. But it is actually a linear function.

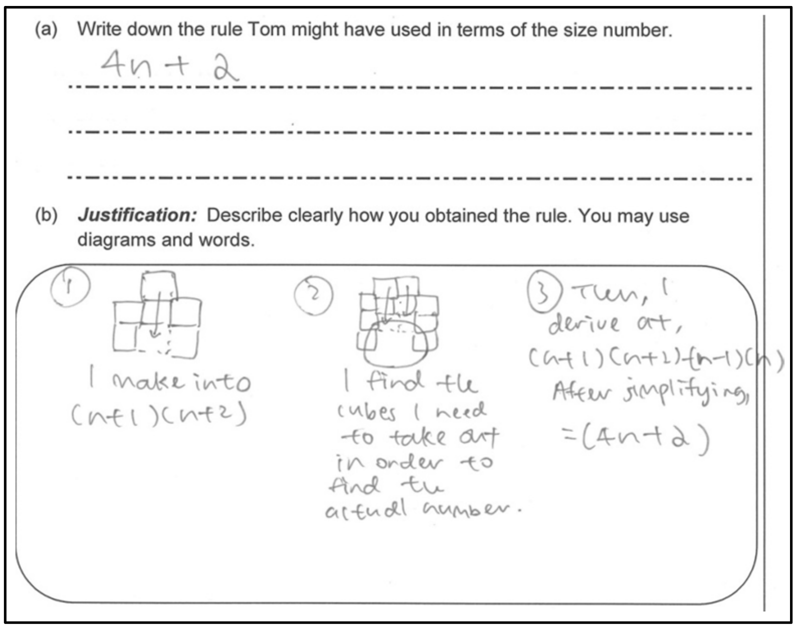


Figure 4.2. Visual representation of $(n + 1)(n + 2) - (n - 1)n$

Towers contained two further unusual rules: $(n + 2)^2 - n^2 - 2$ (Code 611) and $4\left(n + \frac{1}{2}\right)$ (Code 616). Like $(n + 1)(n + 2) - (n - 1)n$ described above, $(n + 2)^2 - n^2 - 2$ is another superficial quadratic disguise for the underlying linear rule. For the second rule, $4\left(n + \frac{1}{2}\right)$, what is surprising is the existence of a fraction in the expression, which would unlikely have any geometrical significance if it was to be explained pictorially. This is because the rule was determined through mere guessing.

A couple of rules spotted in quadratic generalising tasks such as *Oh Deer!* and *Tulips* were equally worth highlighting. One of them is $n^2 + 8 + 3(n - 2)$ (Code 209) from

Oh Deer!, as shown in Figure 4.3 below. It was apparently generated through mere guessing and certain terms in the expression were thus not likely to have any geometrical significance. The other rule is $n + 2[n + (n - 1) + (n - 2) + \cdots + 3 + 2 + 1]$ (Code 404) in *Tulips*. This type of rules is not uncommon in the literature and Steele (2008) has reported a similar case. Although the Code 404 rule describes the structure underpinning the pattern, it is not *algebraically useful*, in Lee's (1996) language. This is because, unlike an algebraically useful rule, it does not allow the direct computation of the output when given an input.

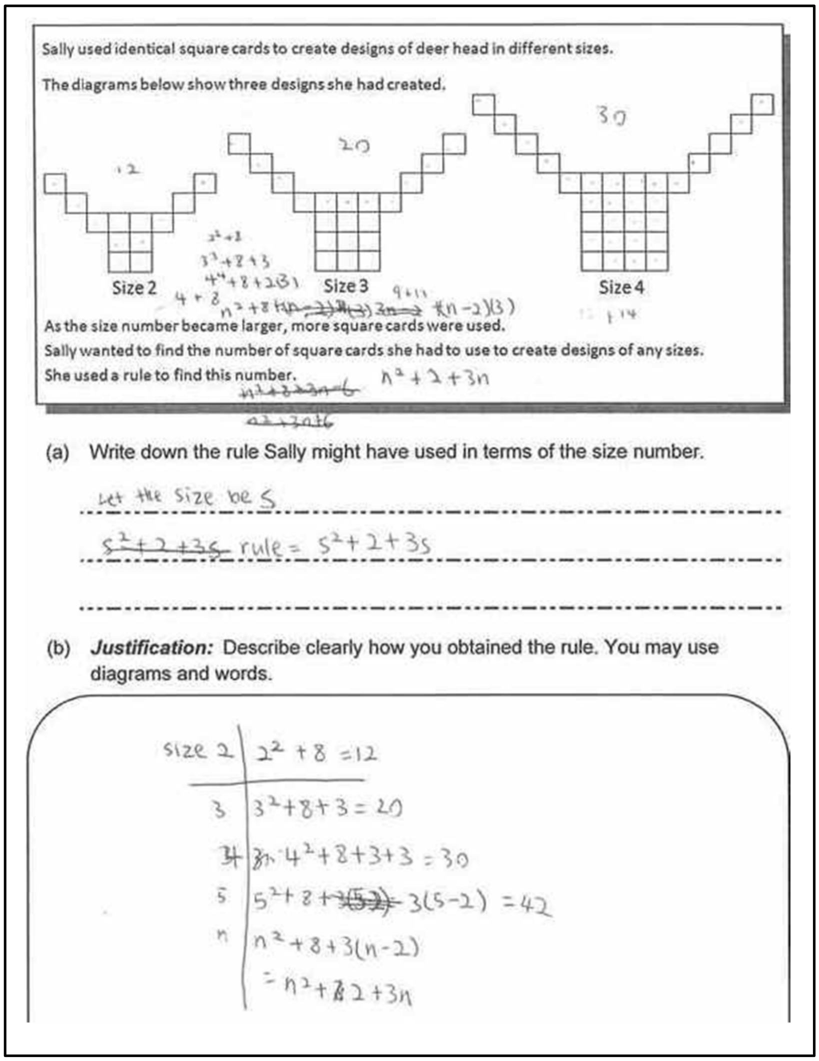
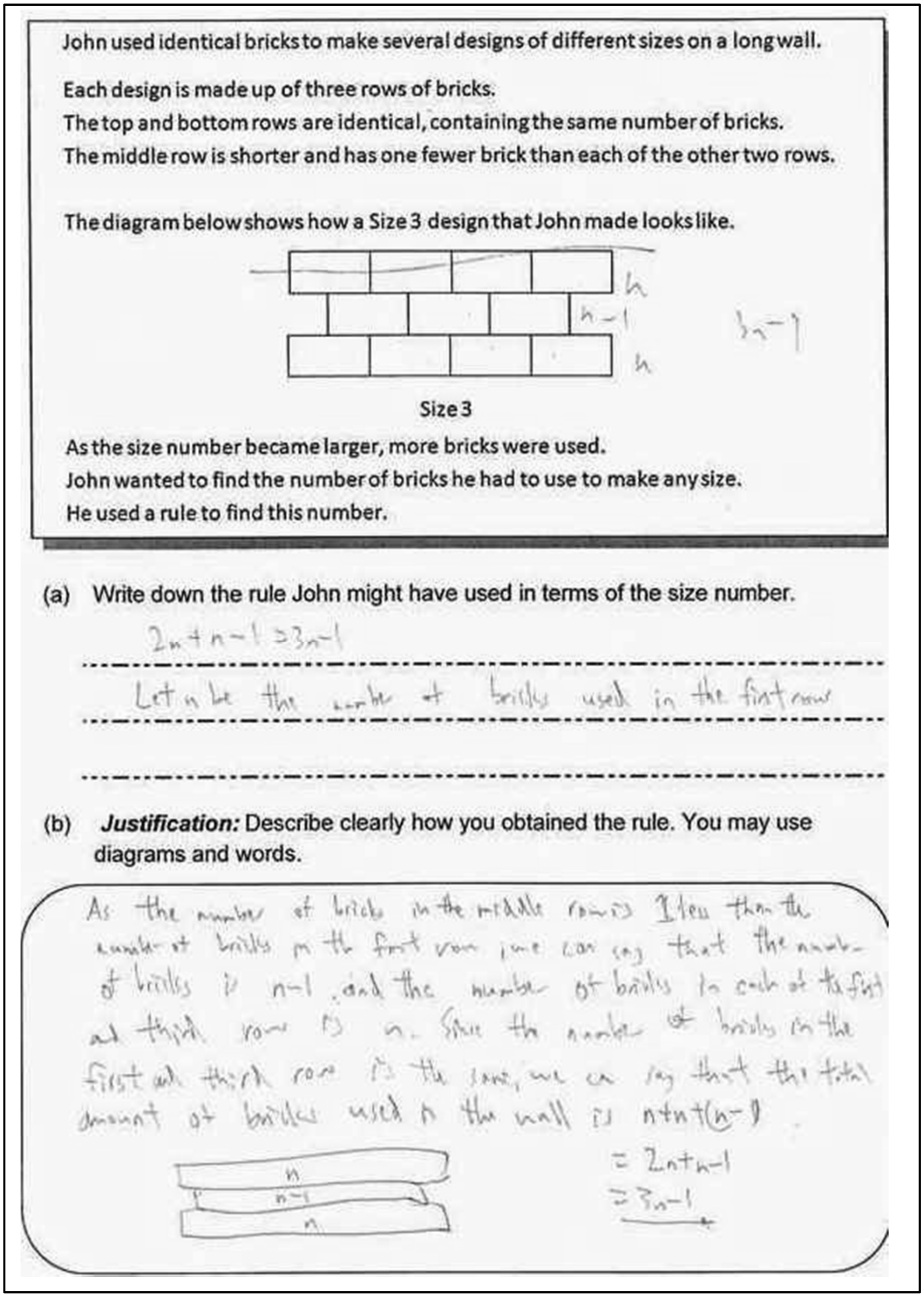


Figure 4.3. A rule from Oh Deer!

So far, the discussion above has focussed on the functional rules that were formulated rightly, but faulty functional rules were created as a result of students relating the size number wrongly to the configuration or making an erroneous assumption of how a pattern would grow with increasing size number. The expressions, $2n + n - 1$ (Code 950) in *Bricks* and $5 + 7(n - 1)$ (Code 980) in *Christmas Party Decorations*, presented in Figures 4.4(a) and 4.4(b) respectively below, are two excellent examples.

In Figure 4.4(a), Student 91M3 labelled n as the number of bricks in the top row of the configuration and by doing so, the bottom row had *n* bricks and the middle row had one brick fewer than the top row: that is, $n - 1$ bricks when expressed in notation. Putting all the three rows together, a general expression for the number of bricks in Size *n* was, therefore, $2n + n - 1$. This expression was clearly wrong because the top row of any configuration did not have the same number of bricks as its size number *n*. In fact it was the middle row that always had the same number of bricks as *n*. This is why *n* should have been linked to the number of bricks in the middle row rather than the top row.

In Figure 4.4(b), Student 25M2 had somehow computed the difference between the given Sizes 1 and 4, then assumed that the difference would be evenly divided over four successive configurations. This discovery eventually led to working out the number of square cards in Sizes 2 and 3, resulting in the linear sequence [5, 12, 19, 26, …]. Through inductive reasoning, the student found an expression for the x^{th} term of the sequence, $5 + 7(x - 1)$. This expression, although it did represent the linear sequence correctly, did not synchronise with the given figural pattern, which underpinned a quadratic relationship. The error would have been detected very easily and avoided if the student had drawn out the configurations for Sizes 2 and 3 and verified his or her response carefully.


John used identical bricks to make several designs of different sizes on a long wall.

Each design is made up of three rows of bricks.
The top and bottom rows are identical, containing the same number of bricks.
The middle row is shorter and has one fewer brick than each of the other two rows.

The diagram below shows how a Size 3 design that John made looks like.

Size 3

As the size number became larger, more bricks were used.
John wanted to find the number of bricks he had to use to make any size.
He used a rule to find this number.

(a) Write down the rule John might have used in terms of the size number.

$2n+n-1=3n-1$

Let n be the number of bricks used in the first row

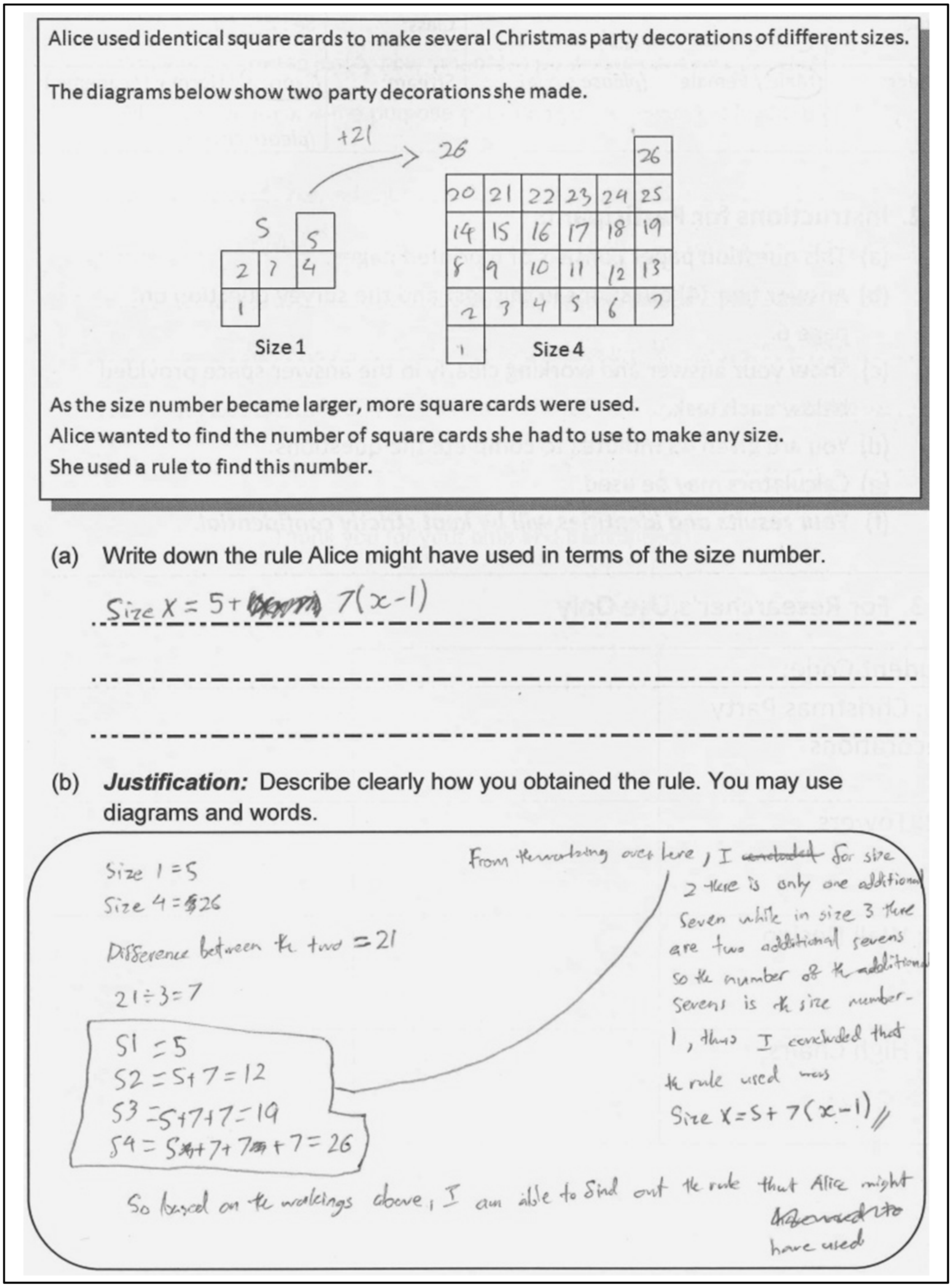
(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

As the number of bricks in the middle row is 1 less than the number of bricks in the first row, we can say that the number of bricks is n-1, and the number of bricks in each of the first and third row is n. Since the number of bricks in the first and third row is the same, we can say that the total amount of bricks used in the wall is $n+n+(n-1)$

$=2n+n-1$

$=3n-1$

(a) Rule for Bricks by Student 91M3



Alice used identical square cards to make several Christmas party decorations of different sizes.

The diagrams below show two party decorations she made.

As the size number became larger, more square cards were used.
Alice wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

(a) Write down the rule Alice might have used in terms of the size number.

Size X = 5 + 7(x-1)

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Size 1 = 5
Size 4 = 26
Difference between the two = 21
21 ÷ 3 = 7

S1 = 5
S2 = 5+7 = 12
S3 = 5+7+7 = 19
S4 = 5+7+7+7 = 26

From the workings over here, I ~~concluded~~ for size 2 there is only one additional seven while in size 3 there are two additional sevens so the number of the additional sevens is the size number - 1, thus I concluded that the rule used was Size X = 5 + 7(x-1)

So based on the workings above, I am able to find out the rule that Alice might ~~have used to~~ have used

(b) Rule for Christmas Party Decorations by Student 25M2

Figure 4.4. Wrong rules produced by students

The discussion will now turn to examine another common incorrect rule – the recursive rule, which was seen in a considerable number of student scripts. As pointed out previously, every *JuStraGen* task asked for a functional rule, thus the recursive rule was deemed as a wrong response. Drawing results from Table 4.10 and Table 4.12, the number of Express students supplying a recursive rule for the linear generalising tasks (Codes 120, 320, 620 and 820) varied between 36 (11%) and 47 (14%) out of 337 students but the numbers of Normal (Academic) students surged to 69 (39%) and peaked at 82 (46%) out of 178 students. The low percentages of Express students suggest that these students had generally demonstrated a reasonable understanding of the task requirements, something which many Normal (Academic) students did not appreciate, as the high proportion of recursive rules indicated clearly. A typical exemplification of the recursive rule, selected from *Bricks*, is illustrated in Figure 4.5 below.

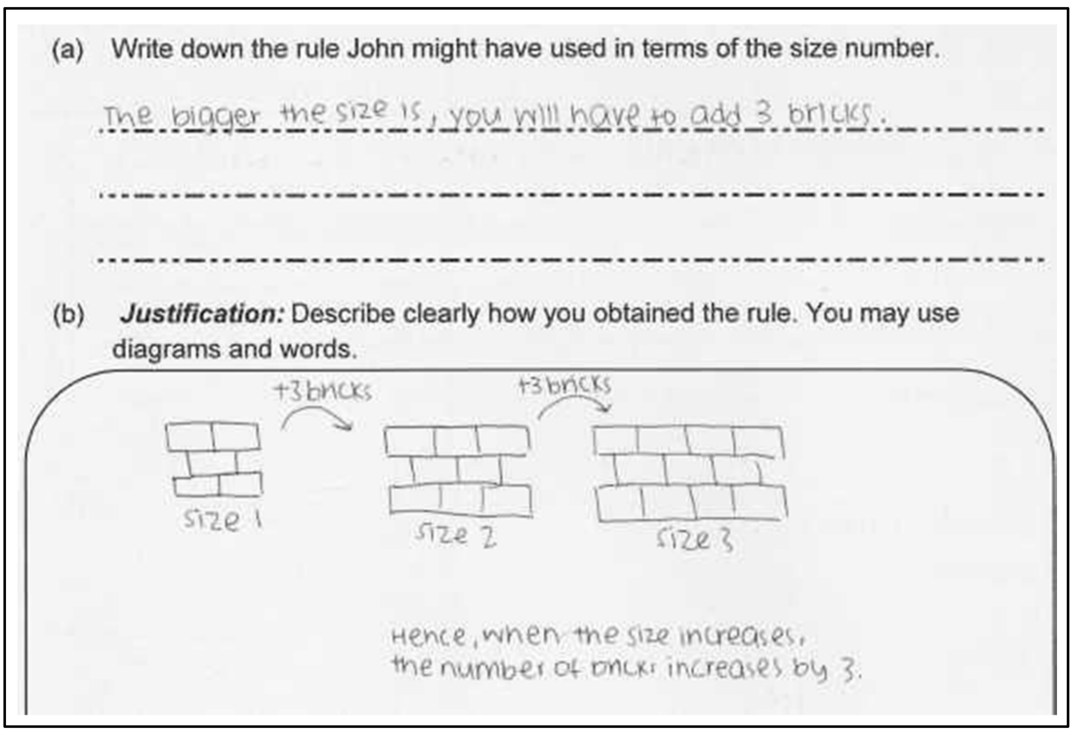
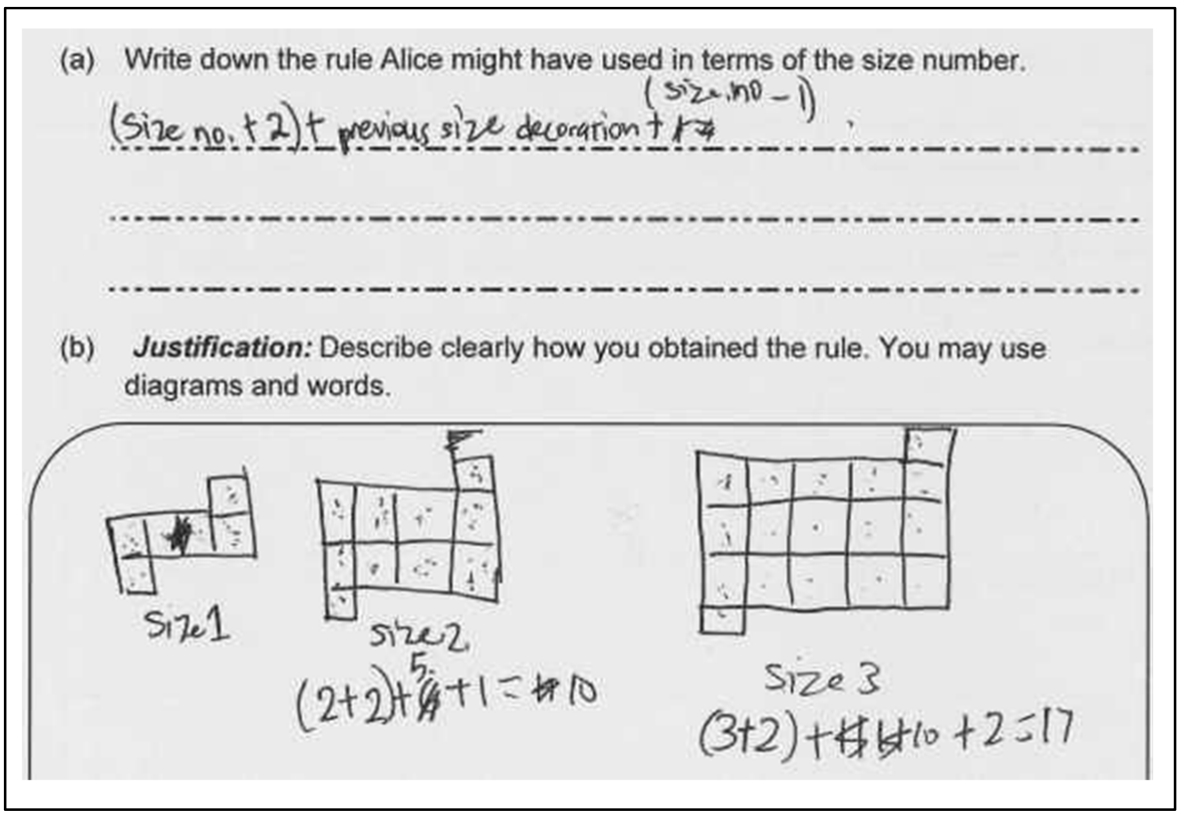


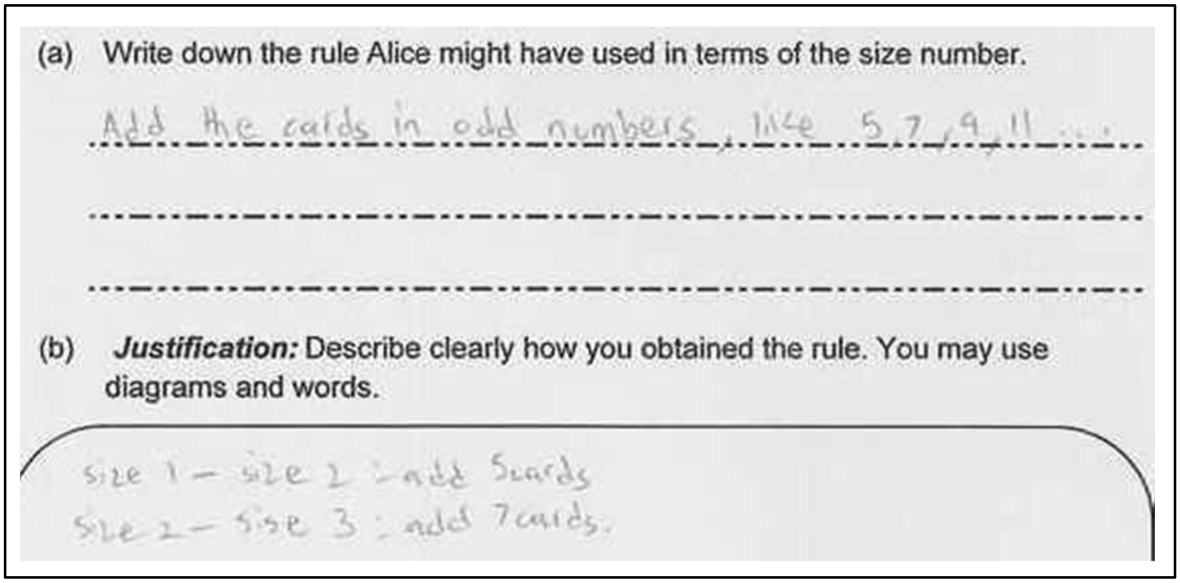
Figure 4.5. A recursive rule for *Bricks*

Comparing with the number of students giving a linear recursive rule, the number of students giving a quadratic recursive rule (Codes 220, 420, 520 and 720) was very much lower. Based on the results in Table 4.11 and Table 4.13, the percentages of students dipped to between 13 (4%) and 22 (7%) in the Express course, and plummeted to between 3 (2%) and 18 (10%) in the Normal (Academic) course. These small figures were not surprising since the recursive rule for a quadratic pattern was harder to articulate than the one for a linear pattern. In view of this difficulty, student responses that listed *only* the first differences correctly were viewed as recursive rules as well. But such responses were coded as 221, 421, 521 or 721 depending on the generalising task to distinguish them from those coded 220, 420, 520 or 720. Two examples of recursive rules taken from the *Christmas Party Decorations* are offered in Figure 4.6 below, with the rule coded 520 in (a), and the rule coded 521 in (b).

Figure 4.6(a) demonstrates the student's attempt to express the quadratic rule as a recursive formula, which, if written properly using notations, is $T_n = (n+2) + T_{n-1} + (n-1)$ or $T_n = T_{n-1} + (2n+1)$ after simplification. It is believed that this way of expressing the quadratic rule is not normally taught in Singapore secondary schools as quadratic patterns are not commonly featured in the local mathematics textbooks. This is why it was surprising to find such a response in the student scripts.



(a) Code 520



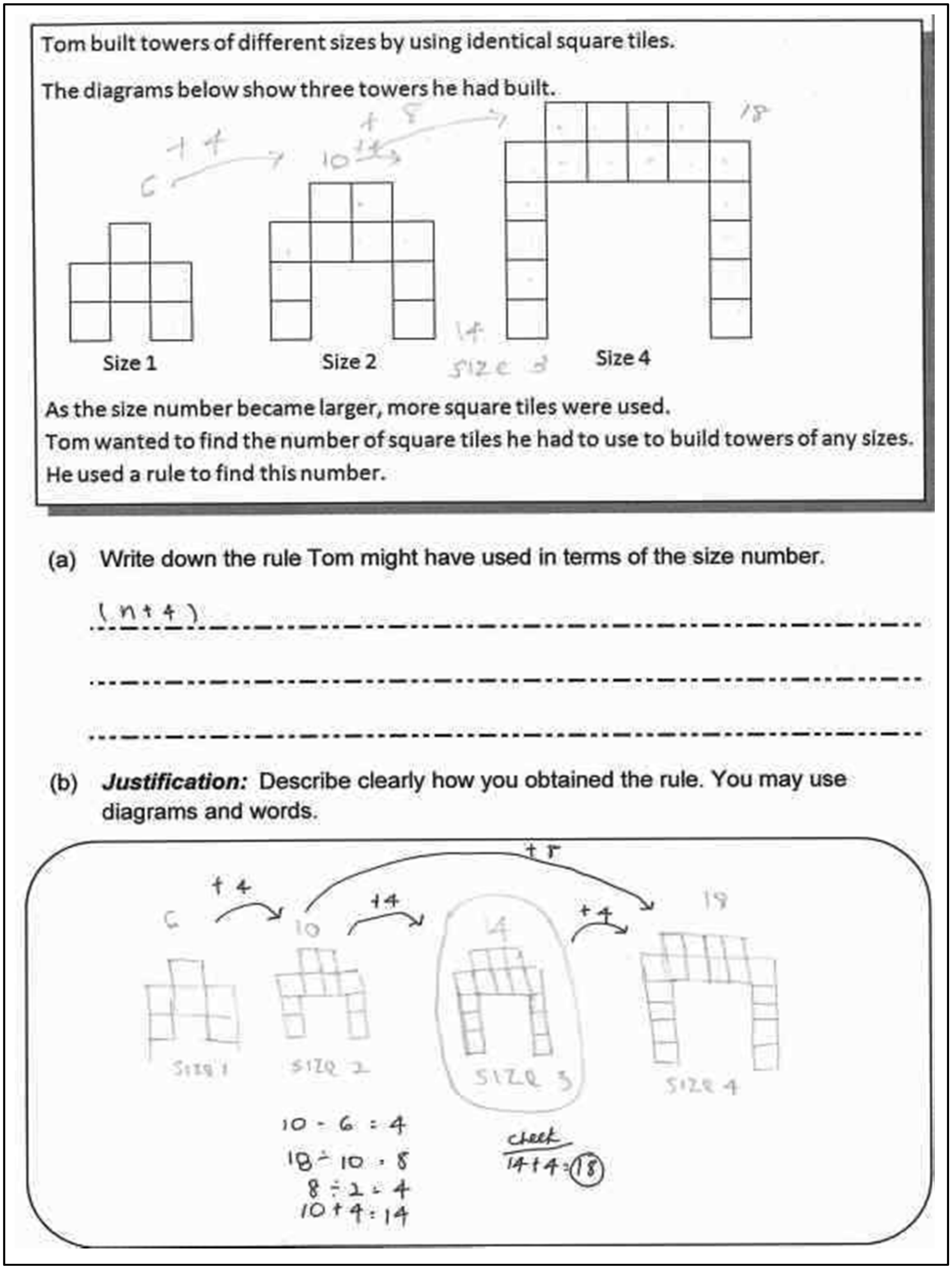
(b) Code 521

Figure 4.6. Recursive rules from *Christmas Party Decorations*

The student response in Figure 4.6(b) above merely indicated the first differences without any further description to say how these differences grew. If the second differences have been mentioned, this response would have been coded as 520.

Another finding about the recursive rules worth mentioning is that their frequencies were contributed largely by G1 students who worked with generalising tasks with successive configurations. Consider *Birthday Party Decorations* for instance, of the 47 Express students who produced a recursive rule, 28 of them came from G1whilst the remaining 19 were from G2, where students dealt with non-successive configurations (see Table 4.10). In *Bricks*, the ratio of Express students in G1 to those in G2 was 5 : 1. The same trend was observed in quadratic generalising tasks as well, with ratios of Express students in G1 to those in G2 equal to 17 : 4 in *Oh Deer!* and 12 : 1 in *Christmas Party Decorations* (see Table 4.11). In very much the same way, the trend continued in the Normal (Academic) course. The ratios of students in G1 to those in G2 were 8 : 3 in *Bricks*, 17 : 6 in *Towers*, 2 : 1 in *Tulips* as well as *Christmas Party Decorations*, and 10 : 3 in *Wall Designs* (see Table 4.12 and Table 4.13).

Finally, the review of Singapore students' performance in the GCE "O" level examinations in Section 2.5.2.1 had found that the expression, $n + k$, where k is a numerical value, was a rather common incorrect answer given as the general rule for the n*th* term. The present study was no exception as such expressions also emerged in all the *JuStraGen* tasks. An example of an incorrect recursive rule for a linear task is offered in Figure 4.7(a) when the student gave $n + 4$ in *Towers*.



Tom built towers of different sizes by using identical square tiles.

The diagrams below show three towers he had built.

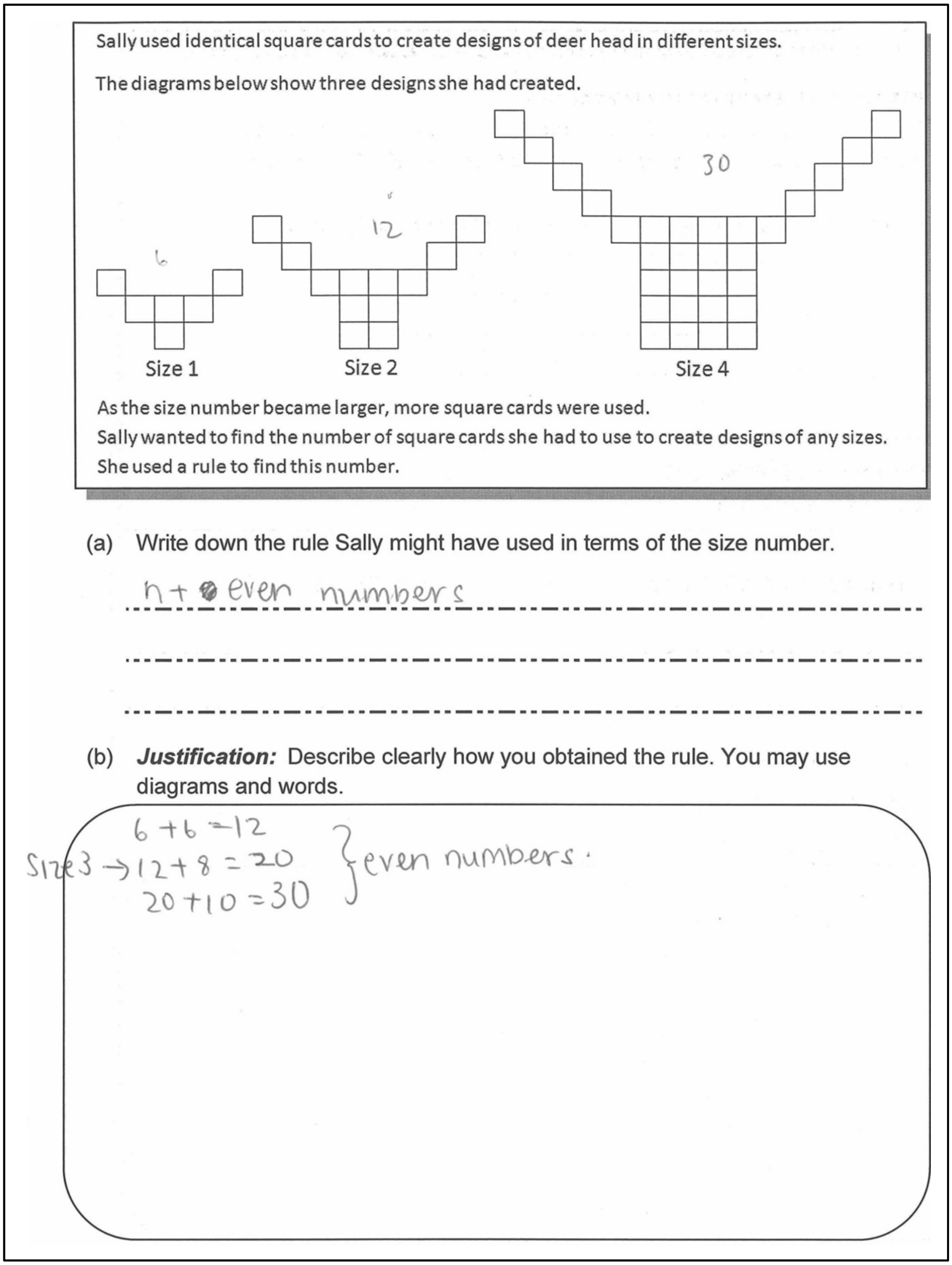
As the size number became larger, more square tiles were used.
Tom wanted to find the number of square tiles he had to use to build towers of any sizes.
He used a rule to find this number.

(a) Write down the rule Tom might have used in terms of the size number.

(n + 4)

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

(a) Linear generalising task: Towers



Sally used identical square cards to create designs of deer head in different sizes.

The diagrams below show three designs she had created.

6 12 30

Size 1 Size 2 Size 4

As the size number became larger, more square cards were used.
Sally wanted to find the number of square cards she had to use to create designs of any sizes.
She used a rule to find this number.

(a) Write down the rule Sally might have used in terms of the size number.

n + even numbers

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

6 + 6 = 12
Size 3 → 12 + 8 = 20
20 + 10 = 30

} even numbers.

(b) Quadratic generalising task: Oh Deer!

Figure 4.7. Incorrect recursive rule

Figure 4.7(b) illustrates an example of a recursive rule, $n + even\, numbers$, for the quadratic task of *Oh Deers!*. However, contrary to the findings in the literature review, the appearance of such rules was rather scarce in the present study. As shown in Table 4.10 to Table 4.13, there were no more than 12 (4%) cases in the Express course and no more than 18 (10%) cases in the Normal (Academic) course.

1.1.2 How do the different equivalent forms of functional rules vary with the different courses the students are enrolled in?

From Table 4.10 to Table 4.13, it is noted that Express students produced an overall of six to 13 different types of equivalent functional rules for each *JuStraGen* task whereas the overall values in the Normal (Academic) course varied between two and six. Figure 4.8 below depicts these two sets of information in graphical form, displaying from left to right the four linear tasks first then followed by the four quadratic tasks.

Comparing the various functional rules established in each task between the two groups of students, the Express students were found to derive more types of equivalent functional rules than their Normal (Academic) counterparts across all *JuStraGen* tasks. For instance, there were 13 different functional rules created in *High Chairs* in the Express course, contrasting with six types in the Normal (Academic) course. Of the four linear tasks, *High Chairs* attained the greatest number of equivalent rules formed in both courses whilst *Birthday Party Decorations* had the least in both courses. For the quadratic tasks, the number of equivalent rules formed by the Express students peaked in *Tulips* but turned out to be the least in the Normal (Academic) course.

Students in both courses shared the most frequently formulated functional rule in all *JuStraGen* tasks, with *Towers* as the only exception, as indicated in Table 4.10 to Table 4.13 above. For instance, $3n + 2$ was detected the most number of times in *Bricks* and in *Birthday Party Decorations* in both the Express and Normal (Academic) student samples. So were $3n + 5$ in *High Chairs* and $n(n + 2)$ in Tulips. However, the outcome was different in *Towers*, with $2(n + 1) + 2n$ having the highest frequency of occurrence in the

Express student sample and the closed form of the rule, $4n + 2$, in the Normal (Academic) student sample.

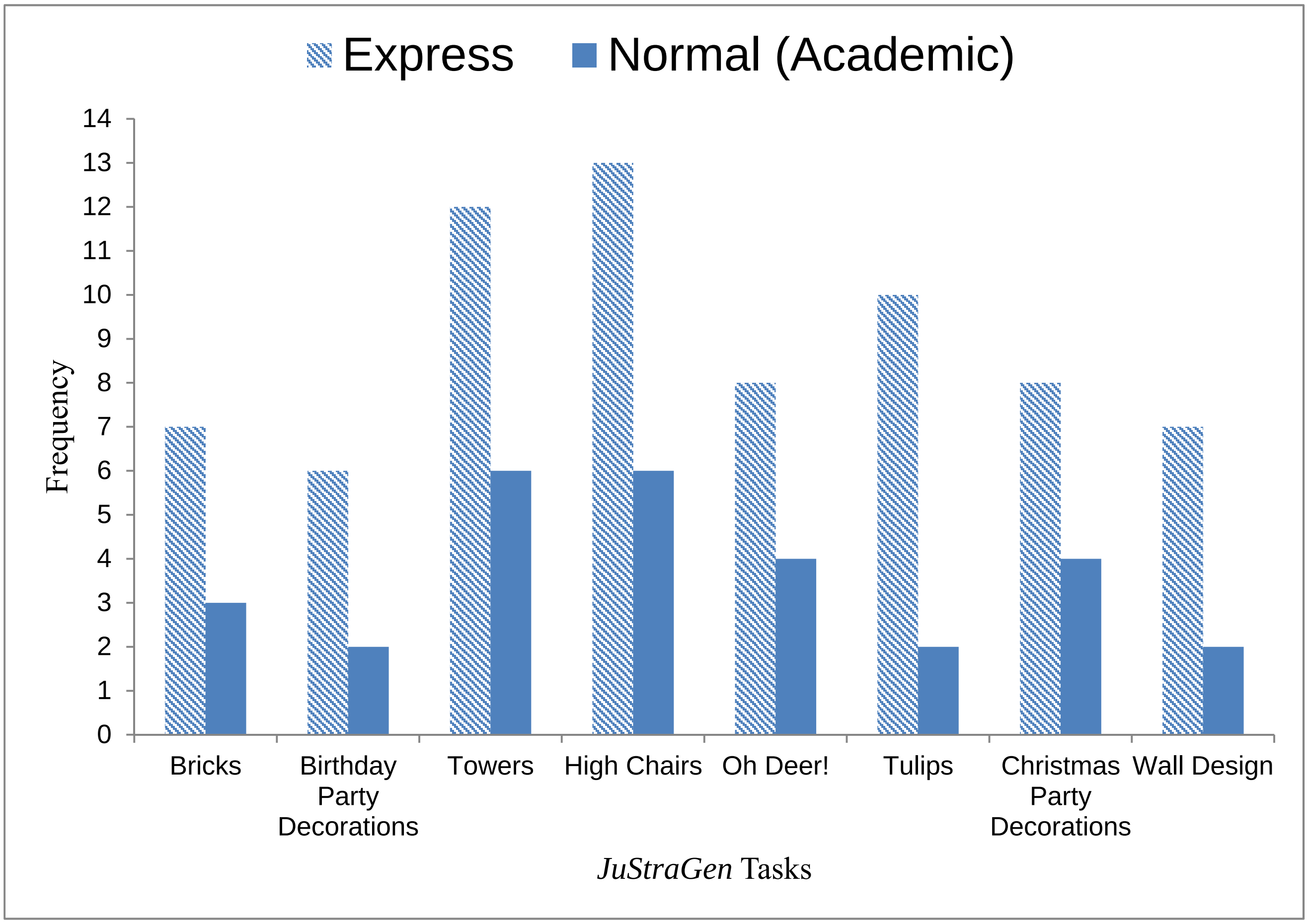


Figure 4.8. Number of different types of equivalent rules by course

Apart from the difference in *Towers* described above, an unexpected dissimilarity between students in the two courses was also noted in the same generalising task. It was the absence of the rule, $6 + 4(n - 1)$ (Code 602) in the Normal (Academic) student sample. This rule can be worked out by expressing each subsequent term of the pattern as the number of *fours* to be added to the first term 6, then realising that this number of *fours* is always one less than the term's position number in the pattern. Such a way of establishing the rule entails inductive reasoning, which is thought to be a familiar approach commonly engaged by students to derive linear rules. This inductive approach was in fact applied in the other three linear generalising tasks, yielding $5 + 3(n - 1)$ in *Bricks* and in *Birthday Party*

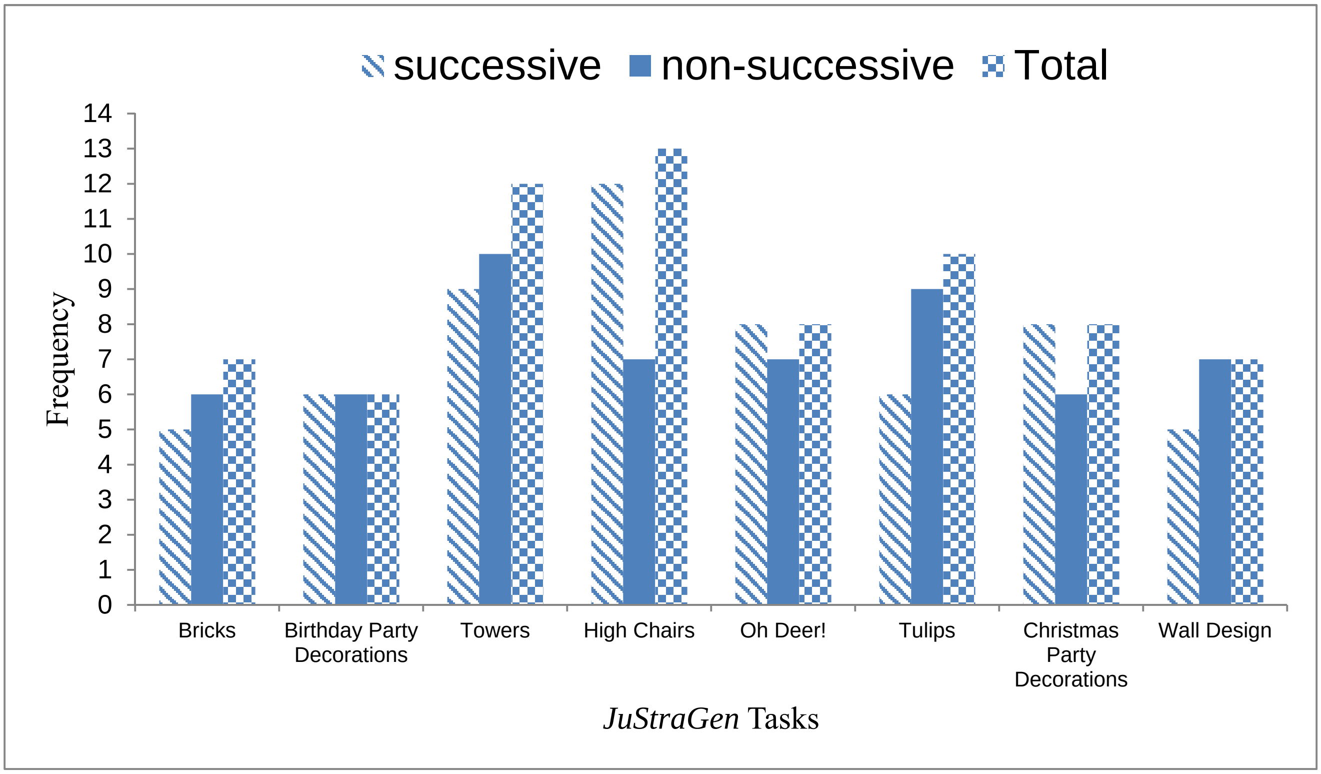
Decorations, and $8{+}3(n-1)$ in *High Chairs*. That is why it was surprising not to find any Normal (Academic) students using that approach to formulate $6+4(n-1)$ in *Towers*.

Another finding emerging from the comparison of the functional rules established by the students in both courses reveals that the Normal (Academic) students tended to produce rules that involve the addition operation. Functional rules involving subtraction were relatively rare, with only two cases in this study. One such rule was $(n+1)^2-1$ (Code 406) in *Tulips* and the other was $3(n+2)-1$ (Code 805) in *High Chairs*. This finding, perhaps, exposes the students' limited knowledge of a variety of appropriate generalising strategies for rule construction.

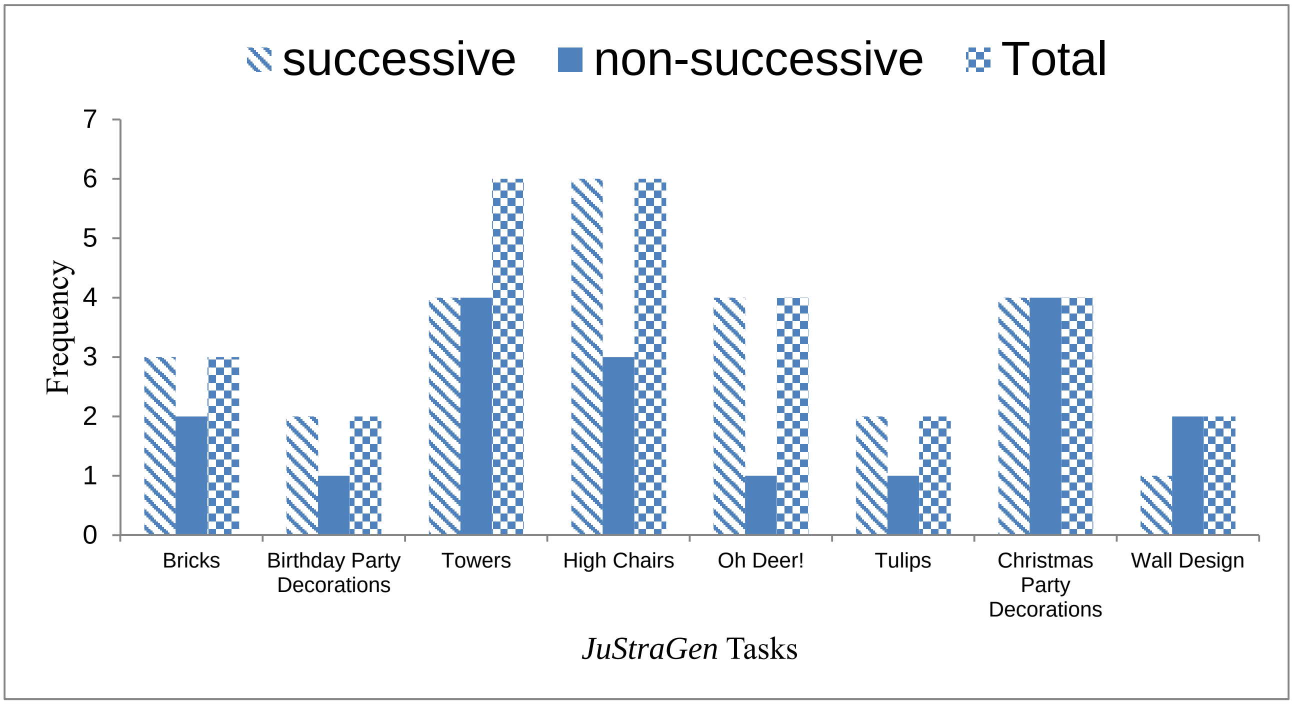
1.1.3 How do the different equivalent forms of functional rules vary with the different formats of pattern display?

The information about the number of equivalent functional rules produced by the Express and Normal (Academic) students in each *JuStraGen* task, extracted from Table 4.10 to Table 4.13, are depicted graphically in Figures 4.9(a) and 4.9(b) respectively. Each graph provides a breakdown of the number of equivalent functional rules by the format of pattern display the students were assigned to. Taking *Tulips* in Figure 4.9(a) for instance, G1 students constructed six different types of equivalent rules whereas G2 students created nine different types. Between them, altogether 10 different types of equivalent rules were identified for this task.

Comparing the number of equivalent functional rules formed by G1 and G2 students in the Express course (see Figure 4.9(a)), G2 students were found to yield more types of functional rules than their G1 counterparts in the following four *JuStraGen* tasks: *Bricks*, *Towers*, *Tulips* and *Wall Design*; but fewer types in *High Chairs*, *Oh Deer!* and *Christmas Party Decorations*. Lastly for *Birthday Party Decorations*, there were as many types of functional rules in G2 as in G1. Amongst the four tasks in which G2 had more types of rules, *Bricks* and *Wall Design* were particularly worth highlighting because the version featuring just a single configuration managed to garner more types of rules than their corresponding successive version. It was therefore quite clear from these findings that generalising tasks with non-successive configurations did not necessarily restrain the Express students' attempts to formulate various types of equivalent functional rules.



(a) Express course



(b) Normal (Academic) course

Figure 4.9. Number of different types of equivalent rules by pattern format

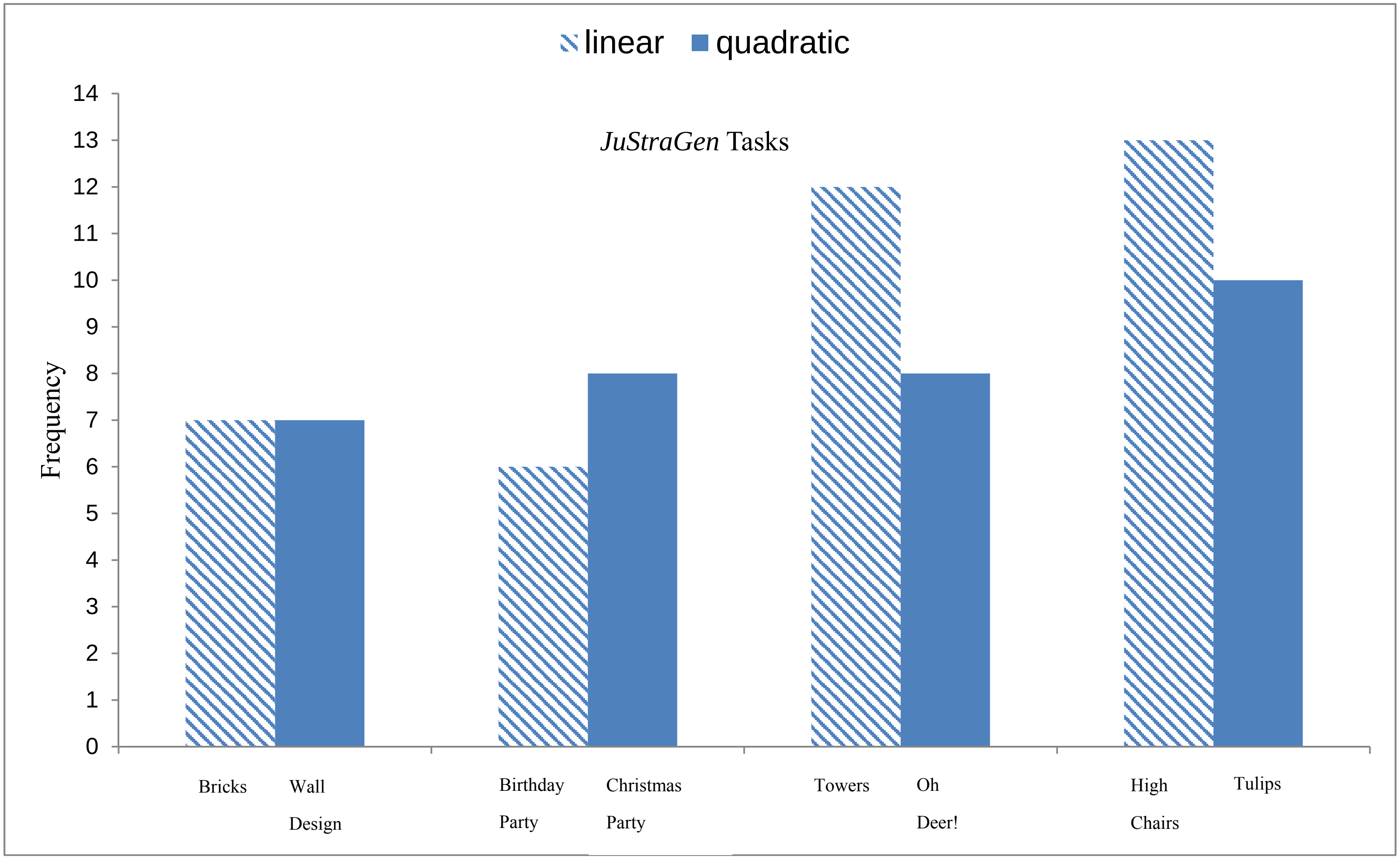
Unlike the performance of Express students in G2, the Normal (Academic) students in G2 produced fewer types of functional rules than their G1 counterparts studying in the same course in most of the generalising tasks, namely, *Bricks*, *Birthday Party Decorations*, *High Chairs*, *Oh Deer!*, and *Tulips* (see Figure 4.9(b)). They overtook the G1 students only in *Wall Design*. In *Towers* and *Christmas Party Decorations*, both groups had a tie in the number of equivalent rules.

A closer examination of all the functional rules reveals an interesting finding about the distribution of the rules between G1 and G2. Quite an appreciable number of the functional rules were found to be common in both G1 and G2, and the remaining rules were seen in only one group but not in the other. For instance, Table 4.10 shows that four (Codes 101, 102, 103 and 105) out of the seven rules in *Bricks* were produced by Express students in both G1 and G2. Of the remaining three rules, two (Codes 106 and 107) were seen in G2 only and the third (Code 108) was seen in G1 only. Those rules in common to both groups of students could have resulted from students applying the same generalising strategies taught by the teachers. A reason why certain rules occurred only in a particular student group in both courses might be that some of the rules might have been motivated by the format of pattern display. Take for instance the rule from *Towers*, $10 + 4(n - 2)$ (Code 607), which was found only in G1 in both Express and Normal (Academic) courses. The *Towers* task given to the G1 students comprised three successive configurations starting with the 10-tile Size 2. Applying the inductive approach to Size 2 would then yield the rule, $10 + 4(n - 2)$. The rule, $11 + 3(n - 2)$ (Code 807), from *High Chairs* is another example seen only in G1.

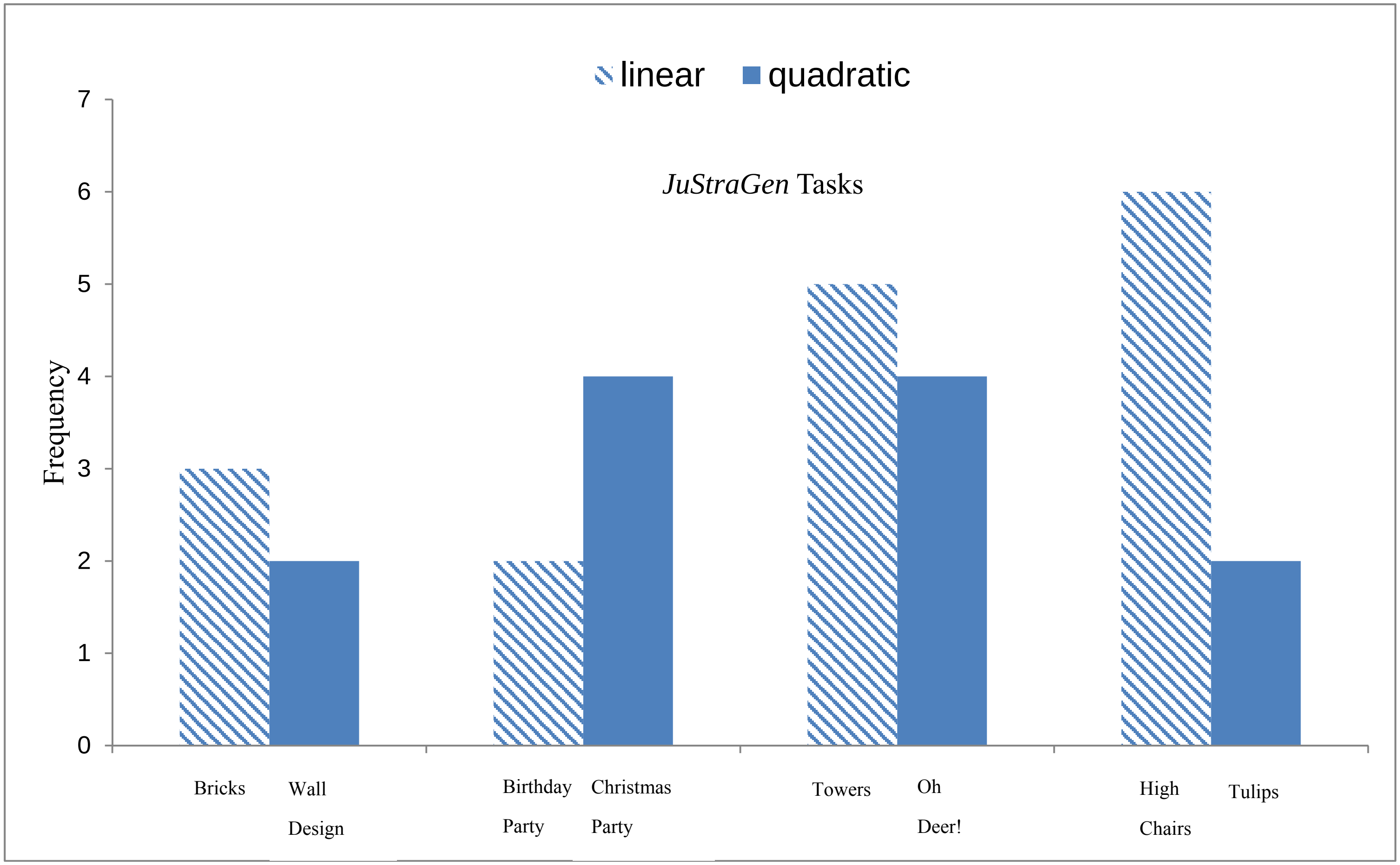
1.1.4 How do the different equivalent forms of functional rules vary with the different types of function?

The two graphs in Figure 4.10 below portray the overall number of different types of equivalent functional rules produced in each linear generalising task alongside the corresponding value in its matching quadratic tasks, with (a) and (b) representing those outcomes in the Express and Normal (Academic) courses respectively. In both courses, the overall numbers in most quadratic generalising tasks were not greater than that in the linear generalising tasks, with *Christmas Party Decorations* as an only exception. The variation in the overall number between each matching pair of linear and quadratic tasks was not much, differing by four at the most in the *Towers – Oh Deer!* pair in the Express course, as well as the *High Chairs – Tulips* pair in the Normal (Academic) course.

From Tables 4.10 to 4.13, it is observed that the most frequent functional rule in each *JuStraGen* task was not always expressed in the closed form. Rules expressed in such a form registered the highest frequency of occurrence only in linear tasks, but not in quadratic tasks. Taking *Birthday Party Decorations* for instance, of the 223 Express students who derived a correct functional rule, 182 of them inferred the closed form, $3n + 2$ (see Table 4.10). Similarly, the same rule was produced by 23 out of 25 successful Normal (Academic) students (see Table 4.12).



(a) Express course



(b) Normal (Academic) course

Figure 4.10. Number of different equivalent rules produced by function type

4.2.1.2 Summary and discussion

Several types of structurally different but equivalent functional rules for both the linear and quadratic tasks were established by the Express students. The wide diversity of quadratic functional rules was especially remarkable for two reasons. First, the quadratic generalising tasks were less commonly featured than linear generalising tasks in the local secondary school mathematics textbooks, yet moderate success rates were achieved for the four quadratic *JuStraGen* tasks. Second, the quadratic rules used in the *JuStraGen* test did not conform to the widely recognised *square* or *triangle* numbers, as seen typically in the local mathematics textbooks or even in the GCE "O" level examinations (see, for example, Cambridge International Examinations, 2006; University of Cambridge Local Examinations Syndicate, 1997). Yet more than half of the students in G1 and in G2 were able to produce a correct functional rule for each quadratic task in the test.

On the other hand, the successes of the Normal (Academic) students were very low. The students tended to focus on the term-to-term difference and write the recursive relationship as the rule for linear tasks. The majority were defeated by the quadratic tasks, giving wrong equations or descriptions of the pattern. Although a sizeable number of students could describe the recursive relationship for linear tasks, not many knew how to do it for quadratic tasks. Compared with their Express counterparts, the smaller number of successful students also engendered fewer ways of visualising the patterns and, hence, a much narrower variety of functional rules. There are several possible explanations for the poor performance of the Normal (Academic) students, one being that many of them are not aware of the kind of rule they are expected to produce for the tasks in the *JuStraGen* test. They do not realise a number pattern can be described in two ways using a recursive relation and a functional relation. So "the rule" in each task refers to the functional and not the recursive relation. Even when the students do understand that they have to write down a rule for finding any term, they often think the recursive rule will suffice. Whilst they are not wrong to think this way since the recursive rule can be used to extend the pattern, they do not fully realise that it cannot compute any term immediately when given its position. Next, some other students may dislike generalising tasks, believing that such tasks require guessing and lack a systematic approach of solving them. Their dislike may have stemmed from their learning experience in school. Finally, given that the students are academically less able and are rarely exposed to quadratic tasks, they stumble and become handicapped when dealing with unfamiliar patterns in the test. Thus many do not even know how to describe the recursive relationship for quadratic tasks.

Most of the functional rules were meaningful in the sense that they could be explained by means of the numerical or figural cues established from the pattern, but a few were apparently not. The wide variety of meaningful functional rules proves that many rules were a result of the students' flexible thinking and discernment of the pattern structure in multiple ways. Although most students had probably been taught certain methods to develop the functional rules, it did appear on the basis of this study that sometimes they preferred to use another method other than the one they had learnt.

One of the incorrect rules, expressed typically as $n + k$, was tremendously infrequent in both Express and Normal (Academic) courses. This result was rather surprising given the perennial GCE "O" Level examiners' reports on the prevalence of such rules in the examinations (see Cambridge International Examinations, 2003a, 2005, 2010a). The small number of students giving this answer in the present study signifies that a handful of students in each course, despite having learnt fundamental algebraic concepts and pattern generalisation prior to taking the *JuStraGen* test, still misunderstood the letter *n* as the previous term of a sequence rather than the position number of the term under consideration. Rivera and Becker (2005) attribute such a student misconception to their lack of notational fluency and competence.

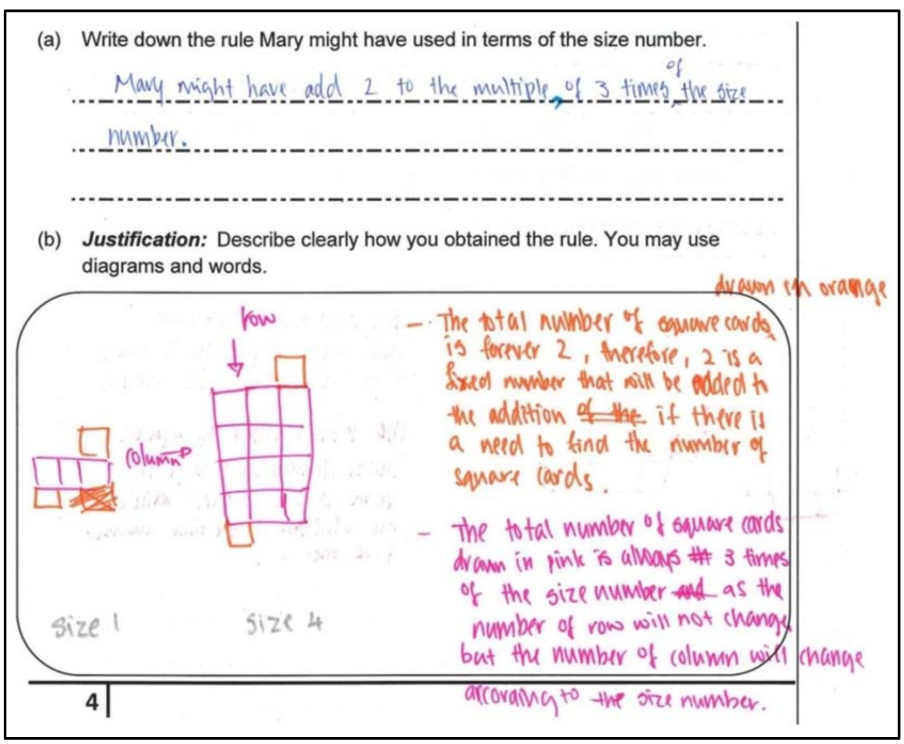
The accomplishment of G2 students in creating more types of different equivalent functional rules in certain *JuStraGen* tasks than G1 students highlights a potential benefit of using generalising tasks involving non-successive configurations. This type of tasks, instead of restraining the students from formulating various types of rules, can seemingly drive them to pursue and attempt different ways of visualising the pattern structure. Next, the lower frequency of recursive rules in G2 when compared to that in G1 offers evidence of another benefit of generalising tasks with non-successive configurations. It appears that such tasks are less likely than those with successive configurations to direct students' attention to the common difference between consecutive terms. With that focus on the term-to-term relationship taken away, the students can then channel their attention to the pattern structure in search for the part that remains invariant and the part that is growing in order to establish the term-to-position relationship.

4.2.1.3 Modalities of rules

This section seeks to characterise the means by which students articulated their functional rules. A comparison was made to study the difference in rule modality between two groups of students studying in different courses, between two sets of generalising tasks with different pattern formats, as well as between two sets of generalising tasks involving different types of function.

1.2.1 What is the modality of the functional rules that students established?

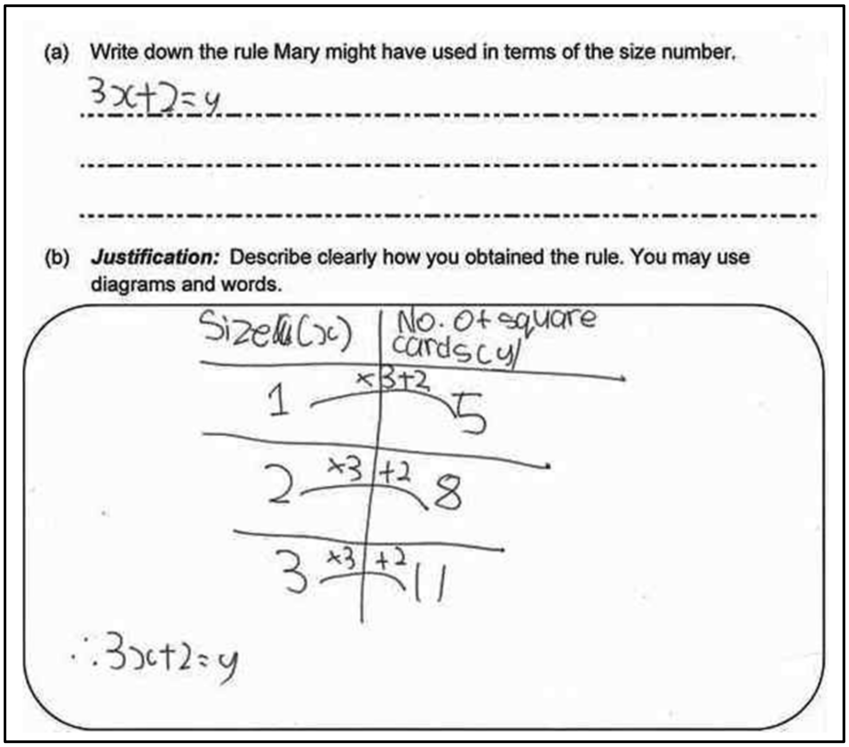
All the correct functional rules established in the *JuStraGen* test were analysed thoroughly to determine their modalities. Three categories of modalities were identified: *in words* (W), *in notations* (N) and *in alphanumeric form* (WN). Figure 4.11 presents three distinct student responses of the same rule, $3n + 2$, for *Birthday Party Decorations*, with (a) in the W mode, (b) in the N mode , and (c) in the WN mode.


(a) Write down the rule Mary might have used in terms of the size number.

Mary might have add 2 to the multiple of 3 times of the size number.

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

(a) Rule expressed in W mode



(a) Write down the rule Mary might have used in terms of the size number.

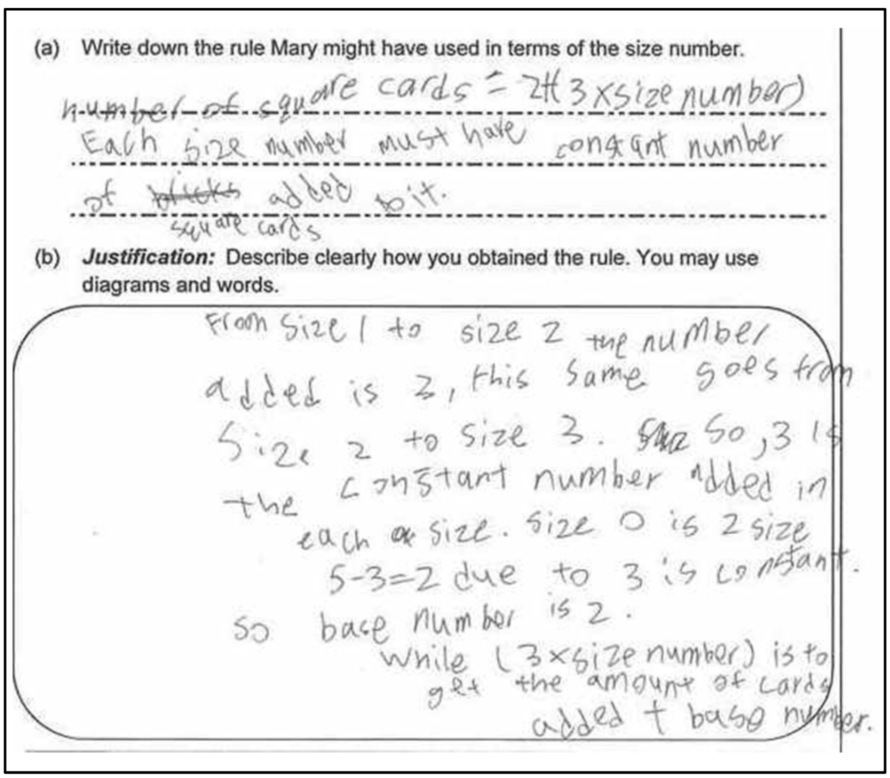
$3x+2=y$

(b) ***Justification:*** **Describe clearly how you obtained the rule. You may use diagrams and words.**

Size (x)	No. of square cards (y)
1 $\xrightarrow{\times 3+2}$	5
2 $\xrightarrow{\times 3\ +2}$	8
3 $\xrightarrow{\times 3\ +2}$	11

$\therefore 3x+2=y$

(b) Rule expressed in N mode



(a) Write down the rule Mary might have used in terms of the size number.

~~number of squ~~are cards = 2+(3×size number)

Each size number must have constant number of ~~blocks~~ square cards added to it.

(b) ***Justification:*** **Describe clearly how you obtained the rule. You may use diagrams and words.**

From size 1 to size 2 the number added is 3, this same goes from size 2 to size 3. ~~Sha~~ So, 3 is the constant number added in each ~~a~~ size. Size 0 is 2 size 5-3=2 due to 3 is constant. So base number is 2. While (3×size number) is to get the amount of cards added + base number.

(c) Rule expressed in WN mode

Figure 4.11. Modalities of functional rule

An overview of the distribution of the various modalities of the functional rules in each *JuStraGen* task by the format of pattern display was provided previously, with the Express students' outcomes presented in Table 4.4 and the Normal (Academic) students' outcomes in Table 4.5. Data will be drawn from these two tables to answer the following three sub-questions.

1.2.2 How does the modality of the students' rules vary with the different courses they are enrolled in?

Referring to the Express students' functional rules for *High Chairs* in Table 4.4, 53% of the G1 students had their functional rules classified under N, 8% under WN and 1% under W. As for students in G2, 60% of them had theirs classified under N, 8% under WN and 1% under W. Drawing data from Table 4.5, the corresponding percentages for the Normal (Academic) students were as follows: 10% N, 5% WN and 2% W from those in G1; 6% N, 3% WN and nobody expressed the rule in words from those in G2. Further examination of the other generalising tasks had found similar results between students in the two different courses. These results reveal that the most significant mode of representing the functional rules was N, followed by WN and W, irrespective of the course the students were studying in.

1.2.3 How does the modality of the students' rules vary with the different formats of pattern display?

As mentioned under sub-question 1.2.2 above, successful Express students in G1 and G2 articulated their *High Chairs* functional rules predominantly in notations, and so did the Normal (Academic) students in the same task. When the data for the remaining *JuStraGen* tasks in Table 4.4 and Table 4.5 were subsequently compared, similar results between the different formats of pattern display again emerged in both courses. Clearly, the frequency of the N mode of representation remained consistently high regardless of the different formats of pattern display.

In addition to the prevalence of symbolic functional rules, the appearance of functional rules in the W mode was very rare amongst students working with non-successive configurations. In fact, this kind of rules was not even seen in a few generalising tasks such as *Birthday Party Decorations*, *Tulips* and *Wall Design*.

1.2.4 How does the modality of the students' rules vary with the different types of function?

Table 4.14 below is another tabular representation of Tables 4.3.1 and 4.3.2, this time reorganising the *JuStraGen* tasks according to their matching pairs. Consider *High Chairs* and *Tulips* for instance. 44% of the Express students in G1 produced a symbolic functional rule in *Tulips* whereas 4% each established their rules in W or WN. Comparing this set of data from *Tulips* with the corresponding set for *High Chairs* reported under sub-question 1.2.2 above, it was evidently clear that the most common modality of the functional rule was N, then followed by WN and finally W. The same trend was observed not only between the two tasks in G2 but in other matching pairs as well. In short, the modality of the functional rules remained prevalently in the N mode regardless of the different types of function.

Table 4.14: Percentage of students in each task by types of rule modality, the pattern format and course

Generalising tasks	Express						Normal (Academic)					
	Successive (G1) (n = 170)			Non-successive (G2) (n = 167)			Successive (G1) (n = 96)			Non-successive (G2) (n = 82)		
	N	WN	W	N	WN	W	N	WN	W	N	WN	W
Bricks	54	9	2	47	6	4	14	2	1	6		1
Wall Design	44	6	2	42	6	2	5	1	1	3	1	
Birthday Party Decorations	58	8	3	53	7	3	14	3	1	9	1	
Christmas Party Decorations	44	5	4	54	5	2		5		6	1	
Towers	58	8	2	61	7	2	14	2	2	5	1	1
Oh Deer!	47	5	3	47	4	2	6	1	1	5	1	
High Chairs	53	8	1	60	8	1	10	5	2	6	3	
Tulips	44	4	4	46	6	4	8			1	1	

4.2.1.4 Summary and discussion

Three modes of representing a functional rule had been identified in this study, the most common being expressing the rule in notations even though the expected modality of the rule was not mentioned specifically in the *JuStraGen* tasks. This was then followed by expressing the rule in alphanumeric form and finally in words. The large number of students formulating symbolic functional rules across the different courses they were enrolled in, the different formats of pattern display and the different types of functions indicates that many students had attained what Rivera and Becker (2005) called *notational fluency* as well as competence in expressing generality using what Hoyles *et al*. (2009) described as the highly specific mathematical language of algebra. These findings differ from previous results reported by Stacey and MacGregor (2001) who observed that Years 7 to 10 Australian students tended to describe their functional rules in words, and were "often reluctant or unable to write [them] as [equations]" (p. 146).

4.2.1.5 Types of generalising strategies used

Under sub-question 1.1.1, empirical evidence was presented to show that the majority of students in the present study produced two types of rules, namely, recursive and functional. This section seeks to illuminate the generalising strategies that students used to develop the functional rules. The workings of all correct functional rules underwent a meticulous analysis to identify the common generalising strategies used. Several types of strategies were determined and these were then classified by type. Given the establishment of these strategy types, a comparison then followed to probe the variation of strategy types across different student courses, pattern formats and function types. The strategy types and the results of the comparison are discussed below.

> 1.3.1 What are the students' generalising strategies for establishing the functional rule?

The generalising strategies that students used to construct the correct equivalent functional rules fell into four categories: *numerical* (N), *figural* (F), *guess-and-check* (G) and *indeterminate* (I). Table 4.15 shows the frequency of strategy types for each generalising task by the pattern formats and courses whereas Table 4.16 shows the frequency and percentage of the various strategy types for each generalising task by the pattern formats.

Overall, figural generalising strategies (F) was widely used across different pattern formats and student courses in many generalising tasks, including *Towers*, *High Chairs*, *Oh Deer!* and *Christmas Party Decorations*, except for *Wall Design* in which a sizeable number of students in each pattern format and student course chose to use N instead (see Table 4.15). In *Tulips*, a substantial number of Express students favoured G. As can be seen in Figure 4.3, a typical telling sign of guessing lies in the scribbling of some expressions on the script, indicating that the student tested and adjusted them one by one in order to fit the expression to the pattern. Finally, the frequencies of I were high. This happened when the students failed to elaborate clearly on how they arrived at their rules. The student justification in Figure 4.11(b) offers an illustration.

Table 4.15: Frequency of strategy types by pattern formats and courses

		Express								Normal (Academic)							
		Successive (G1) (n = 170)				Non-successive (G2) (n = 167)				Successive (G1) (n = 96)				Non-successive (G2) (n = 82)			
Types of Function	Generalising tasks	N	F	G	I	N	F	G	I	N	F	G	I	N	F	G	I
Linear	Bricks	32	34	8	36	6	56	7	26	2	3	1	10		2	1	3
	Birthday Party Decorations	27	57	5	28	7	75	4	20	2	5	1	9		2	1	5
	Towers	21	64	12	17	7	75	8	27	2	7	4	4		3	1	2
	High Chairs	23	60	8	14	7	82	4	23	1	10	3	2		5	1	1
Quadratic	Wall Design	46	28	8	6	31	23	11	18	4	3			2	1		
	Christmas Party Decorations	1	67	8	14		80	2	20		8	3			3		3
	Oh Deer!	6	56	16	15	4	58	12	14		3	4	1		3		2
	Tulips	9	22	34	23	5	31	29	28	1	1	1	5				2

N: numerical, F: figural, G: guess-and-check, I: indeterminate

Table 4.16: Frequency and percentage of strategy types by pattern formats

	Successive (G1) (n = 266)								Non-successive (G2) (n = 249)							
	N		F		G		I		N		F		G		I	
Tasks	freq	%	freq	%	freq	%	freq	%	freq	%	freq	%	freq	%	freq	%
Bricks	34	13	37	14	9	3	46	17	6	2	58	23	8	3	29	12
Birthday Party Decorations	29	11	62	23	6	2	37	14	7	3	77	31	5	2	25	10
Towers	23	9	71	27	16	6	21	8	7	3	78	31	9	4	29	12
High Chairs	24	9	70	26	11	4	16	6	7	3	87	35	5	2	24	10
Wall Design	50	19	31	12	8	3	6	2	33	13	24	10	11	4	18	7
Christmas Party Decorations	1	1	75	28	11	4	14	5			83	33	2	1	23	9
Oh Deer!	6	2	59	22	20	8	16	6	4	2	61	25	12	5	16	6
Tulips	10	4	23	9	35	13	28	11	5	2	31	12	29	12	30	12

N: numerical, F: figural, G: guess-and-check, I: indeterminate

Tables 4.17 and 4.18 below reveal the types of numerical and figural generalising strategies that students in the present study used to establish the linear and quadratic rules respectively. The numerical category encompasses five strategies types, namely, repeated substitution (Code 12), comparison (Code 13), substituting values into formula (Code 14), finding difference and solving equations (Code 16), and, lastly, grouping (Code 17). The figural category comprises nine strategies types and three of them were the *constructive* strategy (Code 21), the *reconstructive* strategy (Code 23) and the *figure-ground reversal* strategy (Code 24). What is not seen amongst the correct functional rules, however, is the use of the *deconstructive* strategy. Not surprisingly, this finding lends support to a previous finding by Rivera and Becker (2007) who remarked about its rare occurrence in their study. Illustrations of the various numerical and figural strategies will now follow.

Table 4.17: Frequency of types of numerical and figural generalising strategies used in each linear task by pattern formats and courses

		Bricks				Birthday Party Decorations				Towers				High Chairs			
		Express		Normal (Academic)		Express		Normal (Academic)		Express		Normal (Academic)		Express		Normal (Academic)	
Strategy	Sample size	170	167	96	82	170	167	96	82	170	167	96	82	170	167	96	82
Type	Strategies	S	NS	S	NS	S	NS	S	NS	S	NS	S	NS	S	NS	S	NS
Numerical	12	23	4	1		16	6	1		11	5	1		11	6	1	
	13	8	1			10	1			10	2			11	1		
	14	1	1	1		1		1				1					
	16																
	17													1			
	Total	32	6	2		27	7	2		21	7	2		23	7	1	
Figural	21	31	53	3	2	54	70	5	2	63	67	5	3	42	58	6	5
	23										2						
	24					1	2				1				1		
	2113																
	2123																
	2124													3			
	2321	2	2			2	3				3	2		12	13	3	
	2324	1	1							1	2			3	10	1	
	2423																
	Total	34	56	3	2	57	75	5	2	64	75	7	3	60	82	10	5

S: successive, NS: non-successive; Refer to Appendix 8 for the coding schemes for the generalising strategies

Table 4.18: Frequency of types of numerical and figural generalising strategies used in each quadratic task by pattern formats and courses

		Wall Design				Christmas Party Decorations				Oh Deer!				Tulips			
		Express		Normal (Academic)		Express		Normal (Academic)		Express		Normal (Academic)		Express		Normal (Academic)	
Strategy	Sample size	170	167	96	82	170	167	96	82	170	167	96	82	170	167	96	82
Type	Strategies	S	NS	S	NS	S	NS	S	NS	S	NS	S	NS	S	NS	S	NS
Numerical	12																
	13	44	28	4	2												
	14																
	16	1				1					1			1			
	17	1	3							6	3			8	5	1	
	Total	46	31	4	2	1				6	4			9	5	1	
Figural	21	7	10		1	62	69	7	3	49	52	2	3	8	16		
	23	15	9	3						7	6	1		3	5		
	24					1									1		
	2113	2												2	2		
	2123	4	1											1	3		
	2124																
	2321		2			4	7	1						4	1		
	2324						4							4	3	1	
	2423		1														
	Total	28	23	3	1	67	80	8	3	56	58	3	3	22	31	1	

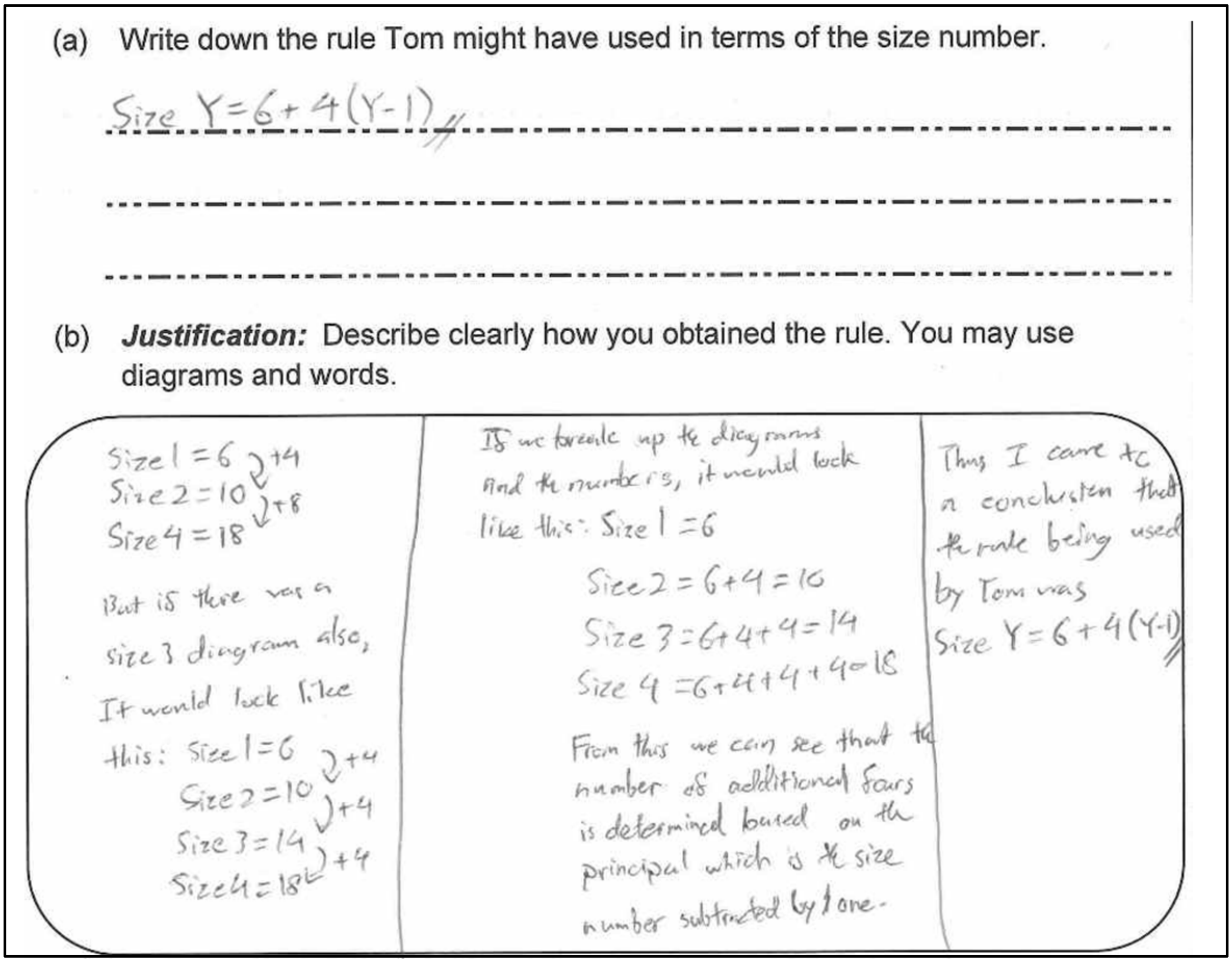
S: successive, NS: non-successive; Refer to Appendix 8 for the coding schemes for the generalising strategies

The student response in Figure 4.11(c) exemplifies one of the five numerical strategies: substituting the "zeroth" term, b, and the common difference, a, into the formula, $an + b$ (Code 14). The remaining four types are described in Figure 4.12 below, with (a) to (d) showing the Codes 12, 13, 16 and 17 types respectively.

The student in Figure 4.12(a) expressed the subsequent terms of the pattern in terms of its first term, 6, and common difference, 4. So the second term was written as 6 + 4, the third term, 6 + 4 + 4, and the fourth term, 6 + 4 + 4 + 4. The number of fours added repeatedly was found to be "*the size number subtracted by one*", as evidenced in the justification. Hence the rule was $6 + 4(y - 1)$, where y denotes the size number.

In Figure 4.12(b), the *Birthday Party Decorations* pattern was first compared with another sequence formed by the multiples of three (the common difference) when the student put "*3 in front and [then] multiply the size number*". After discovering that each term in the latter sequence, whose rule was given by 3(size number), was always two less than the corresponding term of the original sequence, the student then added two to 3(size number) in order to compensate "what is left out". Thus the rule was 3(size number) + 2, in which the student abbreviated the word "number" as "no" in the solution.

Figure 4.12(c) shows a typical approach of determining the quadratic rule. The student first recorded the figural pattern as a sequence of terms, then worked out the first and second differences between the consecutive terms. Next, letting the rule be $an^2 + bn + c$, the student established algebraic expressions for the first three terms of the pattern, as well as the first and second differences in terms of the three unknowns a, b and c. By comparing and equating the algebraic expressions with the numerical terms, the three unknowns were solved, thus yielding the quadratic rule.



(a) Write down the rule Tom might have used in terms of the size number.

Size $Y = 6 + 4(Y-1)$

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Size 1 = 6 +4
Size 2 = 10 +8
Size 4 = 18

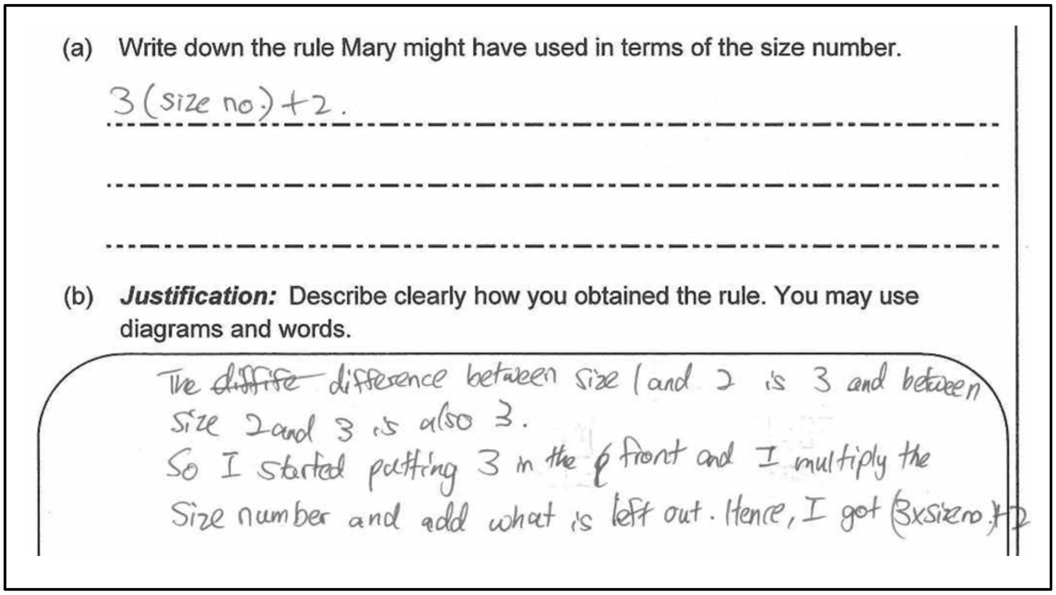
But if there was a size 3 diagram also, It would look like this: Size 1 = 6 +4
Size 2 = 10 +4
Size 3 = 14 +4
Size 4 = 18

If we break up the diagrams and the numbers, it would look like this: Size 1 = 6
Size 2 = 6+4 = 10
Size 3 = 6+4+4 = 14
Size 4 = 6+4+4+4 = 18

From this we can see that the number of additional fours is determined based on the principal which is the size number subtracted by one.

Thus I came to a conclusion that the rule being used by Tom was Size $Y = 6 + 4(Y-1)$

(a) Code–12 strategy in Towers

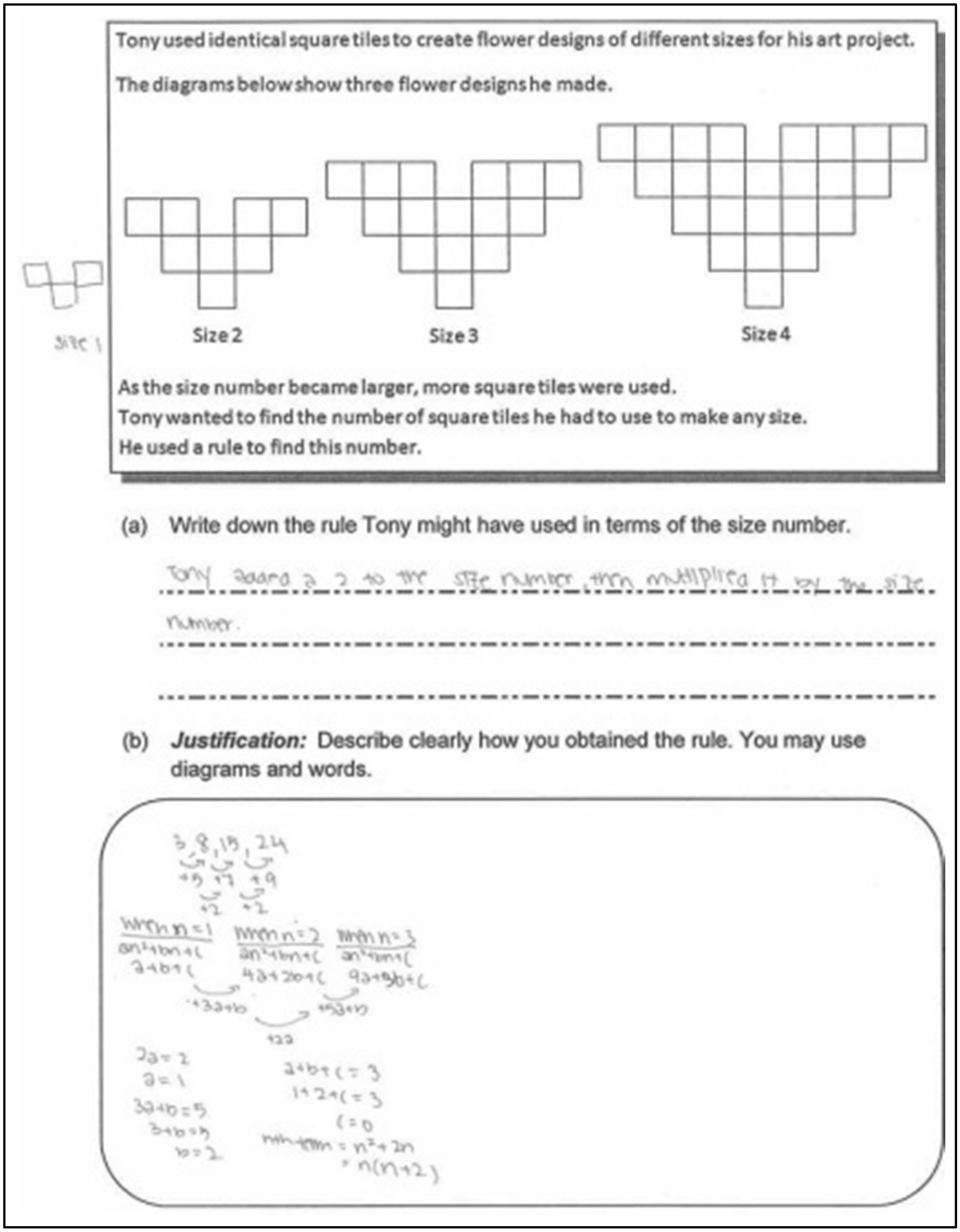

(a) Write down the rule Mary might have used in terms of the size number.

3(size no.)+2.

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

The ~~diffife~~ difference between size 1 and 2 is 3 and between size 2 and 3 is also 3.
So I started putting 3 in the front and I multiply the size number and add what is left out. Hence, I got (3xsize no.)+2

(b) Code–13 strategy in Birthday Party Decorations


Tony used identical square tiles to create flower designs of different sizes for his art project.

The diagrams below show three flower designs he made.

size 1

Size 2

Size 3

Size 4

As the size number became larger, more square tiles were used.
Tony wanted to find the number of square tiles he had to use to make any size.
He used a rule to find this number.

(a) Write down the rule Tony might have used in terms of the size number.

Tony added a 2 to the size number, then multiplied it by the size number.

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

3, 8, 15, 24

+5 +7 +9

+2 +2

when n=1: an^2+bn+c, $a+b+c$

when n=2: an^2+bn+c, $4a+2b+c$

when n=3: an^2+bn+c, $9a+3b+c$

+3a+b +5a+b

+2a

$2a=2$

$a=1$

$3a+b=5$

$3+b=5$

$b=2$

$a+b+c=3$

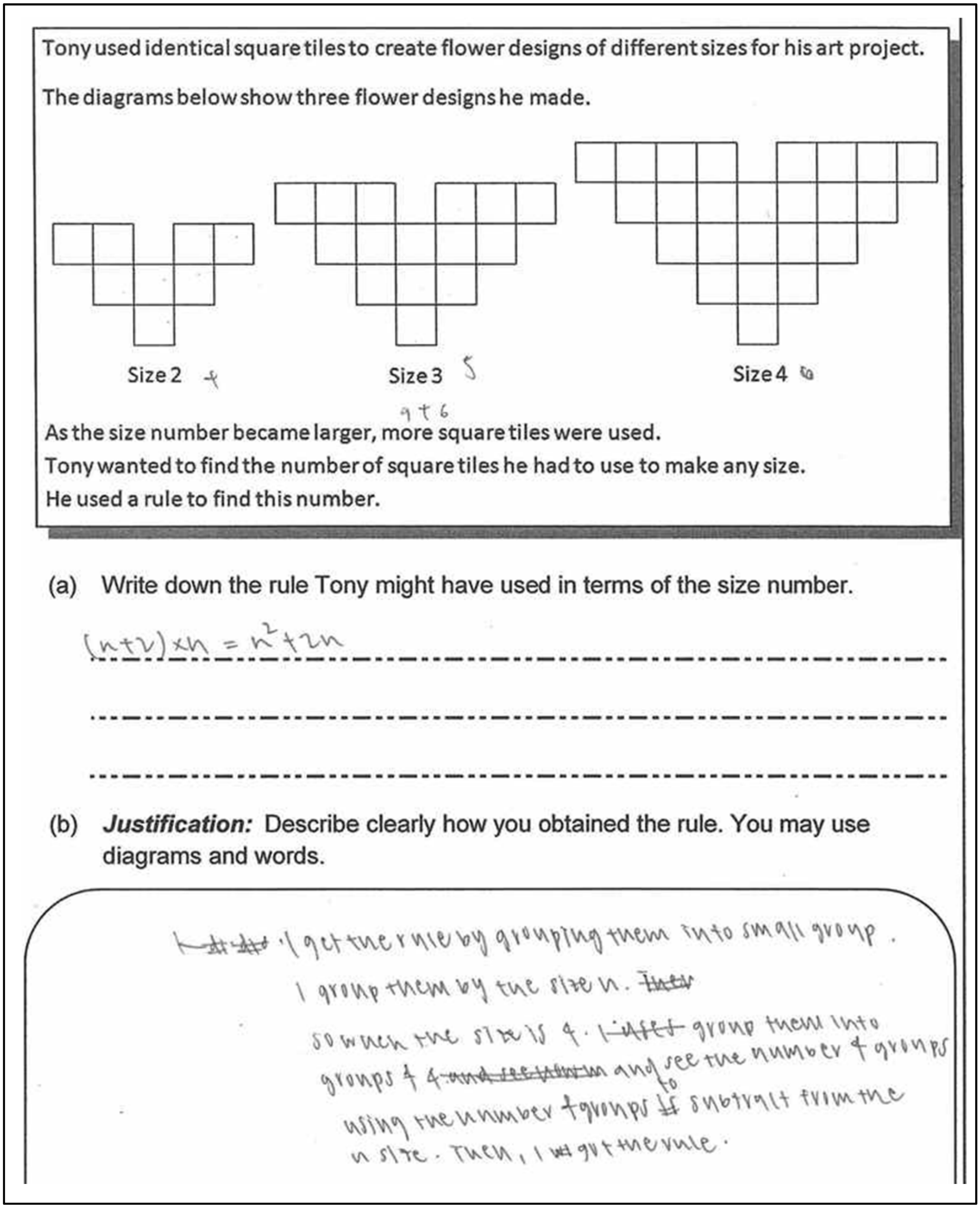
$1+2+c=3$

$c=0$

nth term $= n^2+2n$

$= n(n+2)$

(c) Code–16 strategy in Tulips



Tony used identical square tiles to create flower designs of different sizes for his art project.

The diagrams below show three flower designs he made.

Size 2 4

Size 3 5

Size 4 6

9 + 6

As the size number became larger, more square tiles were used.
Tony wanted to find the number of square tiles he had to use to make any size.
He used a rule to find this number.

(a) Write down the rule Tony might have used in terms of the size number.

$(n+2)\times n = n^2+2n$

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

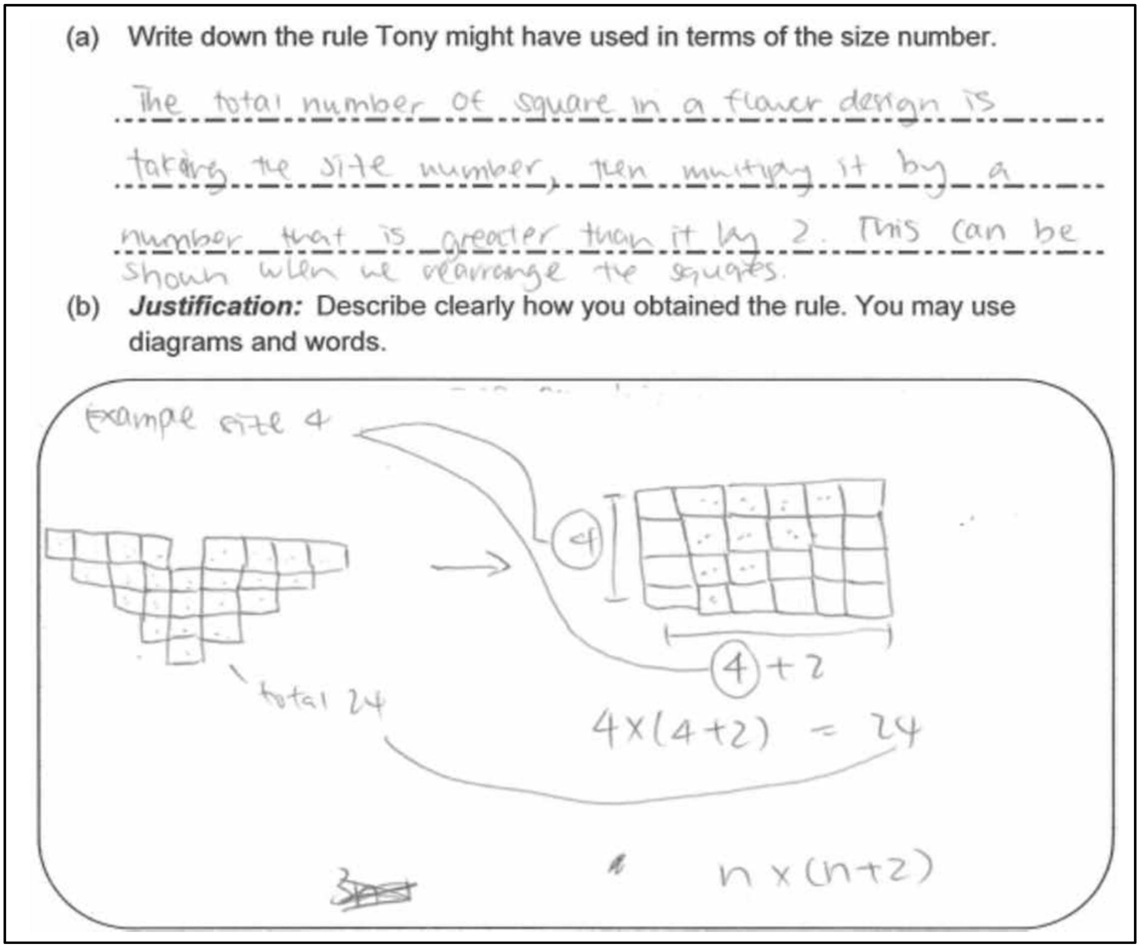
~~I get~~ I get the rule by grouping them into small group.
I group them by the size n. ~~Then~~
So when the size is 4. ~~I after~~ group them into groups of 4 ~~and see how m~~ and see the number of groups using the number of groups to subtract from the n size. Then, I got the rule.

(d) Code–17 strategy in Tulips

Figure 4.12. Numerical strategies

The *grouping* strategy in Figure 4.12(d) is remarkably noteworthy considering it is rarely seen and discussed in the research literature. As seen from the justification and the scribblings below the configurations, the student determined the number of groups of n tiles in each configuration: for instance, there were four groups of two tiles in Size 2, five groups of three tiles in Size 3, and six groups of four tiles in Size 4. Since the number of groups was always two more than the size number, there were, therefore, $(n + 2)$ groups of n tiles in Size n, or a total of $n(n + 2)$ tiles.

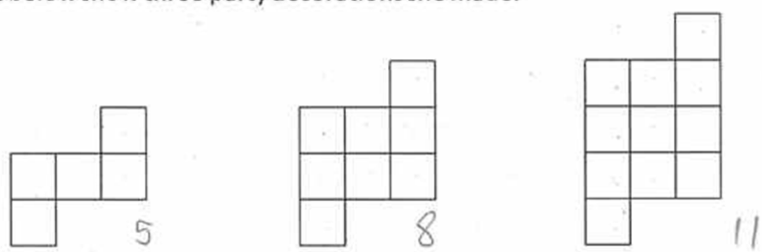
For figural strategies, two examples of the *constructive* strategy were provided earlier in Figures 4.1 and 4.11(a). The *reconstructive* and *figure-ground reversal* types are presented respectively in Figures 4.13(a) and 4.13(b) below. The student using the *reconstructive* strategy rearranged a Size-4 *Tulip* configuration into a 4 by $(4 + 2)$ rectangle, and, after recognising the link between the dimensions and the size number, concluded that the rule was $n(n + 2)$. The other student using the *figure-ground reversal* strategy perceived each *Birthday Party Decorations* configuration as being formed by removing four cards from a "perfect" rectangle comprising $3(n + 2)$ cards. Thus the rule was $3(n + 2) - 4$.



(a) Reconstructive strategy in Tulips

Mary used identical square cards to make several birthday party decorations of different sizes.

The diagrams below show three party decorations she made.



As the size number became larger, more square cards were used.
Mary wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

(a) Write down the rule Mary might have used in terms of the size number.

$T_n = 3(n+2) - 4$

$= 3n + 6 - 4$

$= 3n + 2$

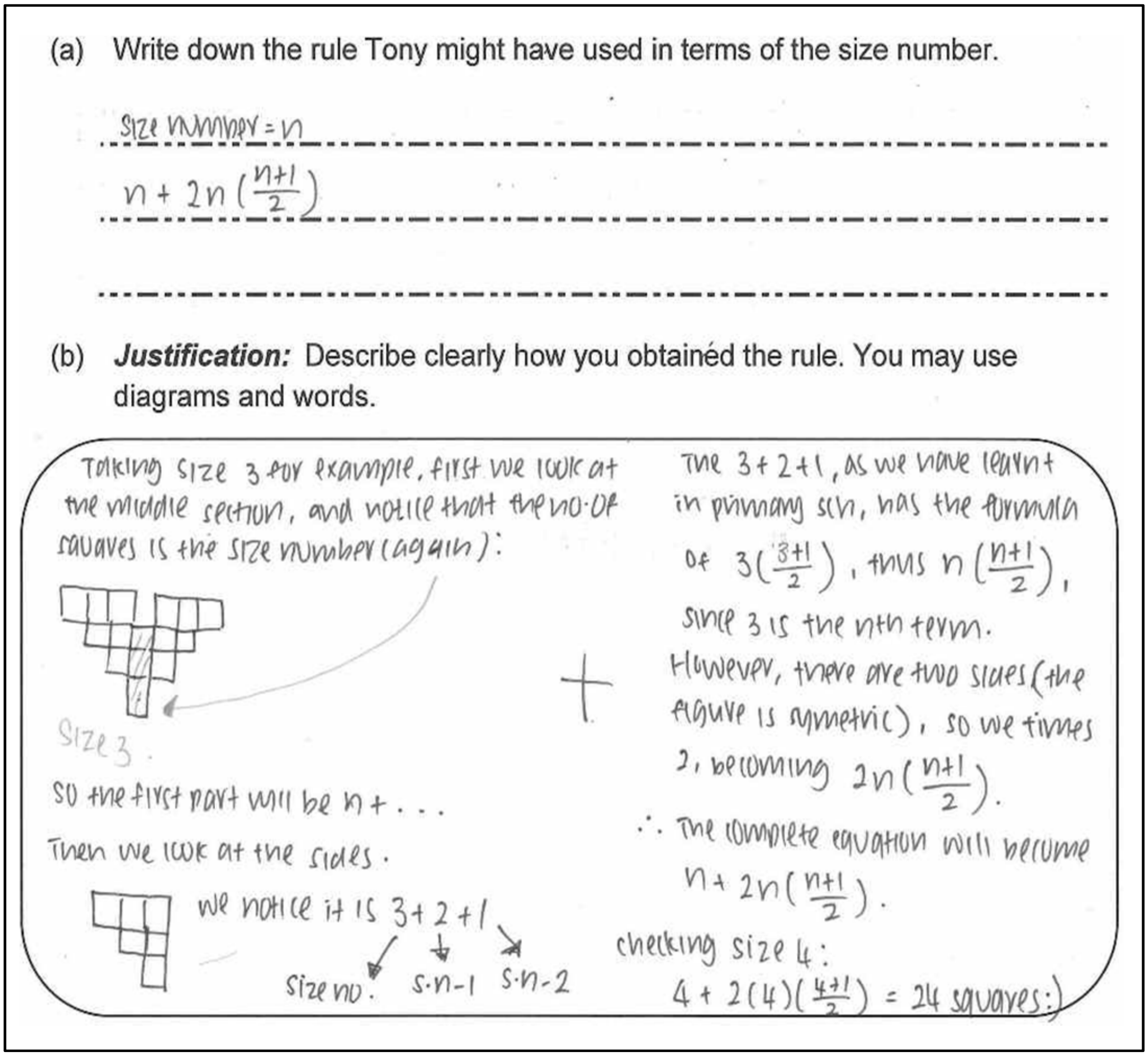
(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Each size had four squares removed from them.
So, if the squares were in one perfect rectangle, it would be 3 times the result of the size number plus 2. Which is $3(n+2)$.
But because it is not a perfect rectangle, 4 has to subtracted from the result. Which is $3(n+2)-4$.
Therefore, the rule is: $3(n+2)-4$
$= 3n+2$

(b) Figure-ground reversal strategy in Birthday Party Decorations

Figure 4.13. Figural strategies

It was noticed in a number of test scripts that some students engaged a *combo* strategy when working out their rules. Six different *combo* strategies were found in the present study and two of them, coded 2113 and 2324, are illustrated in Figure 4.14 below. The example of Code–2113 strategy in (a) shows a student using the *constructive* strategy initially (hence Code 21 were indicated first) to isolate the "stalk" of a tulip from its entire configuration and then relate the number of tiles it was made of to the size number. Subsequently, the student, drawing on the prior knowledge gained in primary school, quoted the correct algebraic expression for the number of tiles in each of the remaining two parts, thus manifesting the application of the *comparison* strategy (Code 13). This Code–2113 strategy is an unusual combination because it encompasses both figural and numerical strategies. A manifestation of the Code–2324 strategy is illustrated in Figure 4.14(b). This example incorporates two separate actions: first, relocating the topmost row of tiles in each configuration to immediately below the second row – exemplifying the reconstructive strategy (Code 23), and second, treating each resulting configuration as a rectangle with missing tiles in the last row – exemplifying the figure-ground reversal strategy (Code 24).



(a) Write down the rule Tony might have used in terms of the size number.

size number = n

$n + 2n\left(\frac{n+1}{2}\right)$

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Taking size 3 for example, first we look at the middle section, and notice that the no. of squares is the size number (again):

size 3

so the first part will be $n + \ldots$

Then we look at the sides.

we notice it is $3 + 2 + 1$

size no.   s.n-1   s.n-2

+

The 3+2+1, as we have learnt in primary sch, has the formula of $3\left(\frac{3+1}{2}\right)$, thus $n\left(\frac{n+1}{2}\right)$, since 3 is the nth term.

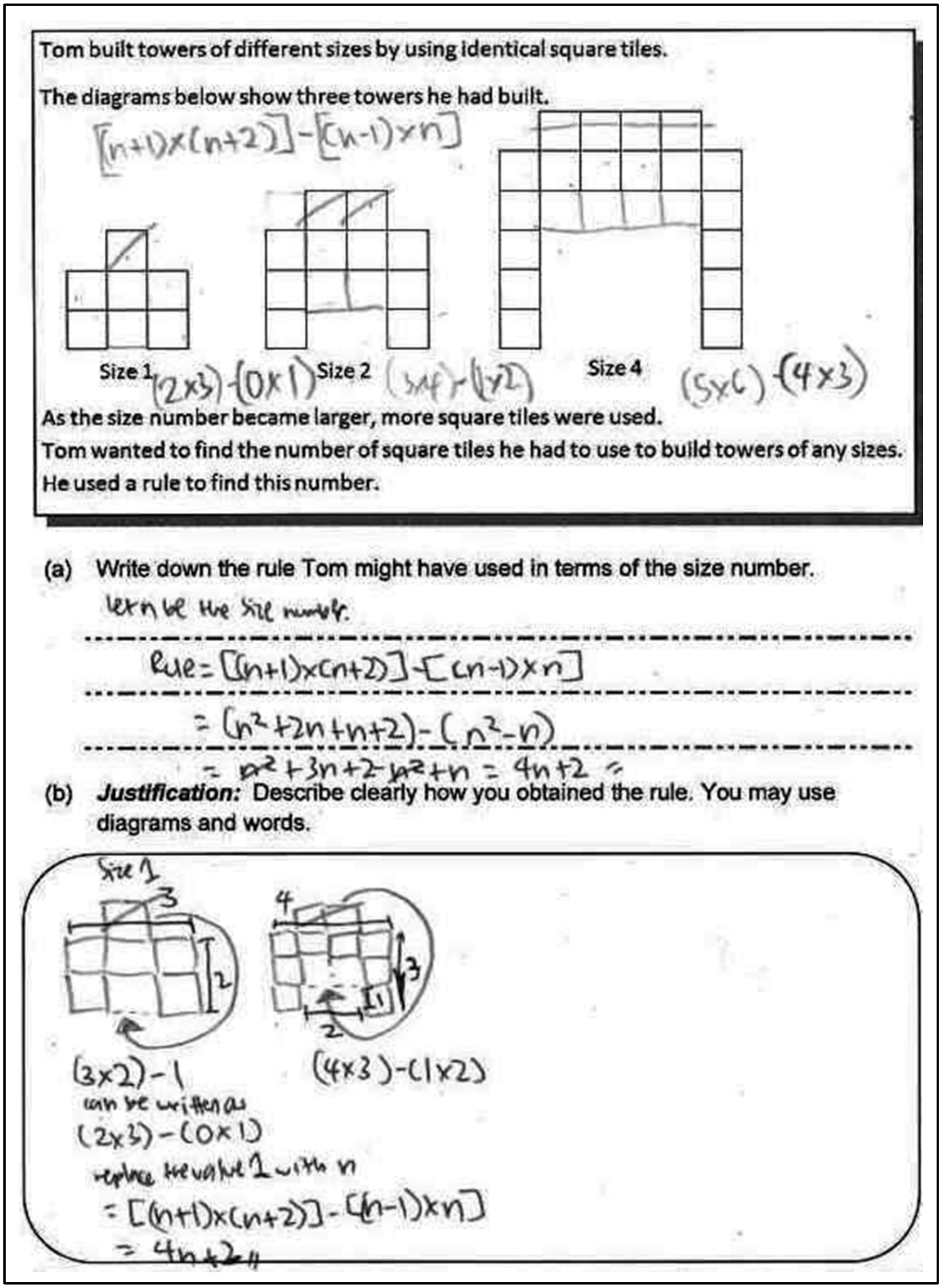
However, there are two sides (the figure is symmetric), so we times 2, becoming $2n\left(\frac{n+1}{2}\right)$.

$\therefore$ The complete equation will become $n + 2n\left(\frac{n+1}{2}\right)$.

checking size 4:

$4 + 2(4)\left(\frac{4+1}{2}\right) = 24$ squares :)

(a) Code–2113 strategy



(b) Code–2324 strategy

Figure 4.14. Combo strategies

1.3.2 How do the students' generalising strategies vary with the different courses they are enrolled in?

Table 4.15 suggests that by and large both Express and Normal (Academic) students preferred figural to numerical generalising strategies when dealing with figural generalising tasks. Some students in the two different courses employed the *guess-and-check* strategy; some did not even describe clearly how their rules were obtained, thus rendering their strategies to be classified as *indeterminate*.

Drawing evidence from Tables 4.17 and 4.18, the Express students appeared to be more well-informed about the various generalising strategies than the Normal (Academic) students. The Express students had demonstrated the use of all the numerical and figural strategies listed in those tables whereas certain strategies such as *finding difference and solving equations* (Code 16) and the *figure-ground reversal* (Code 24) were not observed in the Normal (Academic) students' repertoire of generalising strategies. Two further observations about this repertoire were also noted. First, there were only two strategy types involving a combination of strategies, perhaps implying the students' inexperience with engaging in such options for constructing rules. Next, there were also very few cases of students applying the *repeated substitution* strategy (Code 12), suggesting that students might be unfamiliar with this approach. This is particularly surprising when it was believed to be a commonly used strategy in the Singapore mathematics curriculum.

To work out the linear rules, the Express students relied mainly on the four strategies coded 12, 13, 21 and 2321. For quadratic tasks, the two commonly used strategies were the *constructive* (Code 21) and *reconstructive* types (Code 23). However, the conclusions for the Normal (Academic) students' preference of generalising strategies were far less clear and definite given their low success rates.

1.3.3 How do the students' generalising strategies vary with the different formats of pattern display?

Table 4.16 indicates that figural strategies were rather widely used by the students. In G1, the F category clinched the highest percentage of students in five generalising tasks, namely, *Birthday Party Decorations*, *Towers*, *High Chairs*, *Oh Deer*! and *Christmas Party Decorations*. Of the remaining three tasks, *Bricks* had the most cases of *indeterminate* strategies, *Wall Design* has the most cases of numerical strategies, and *Tulips* had the most cases of guess-and-check. However, in G2, the percentage of F peaked in nearly all the *JuStraGen* tasks, with *Wall Design* as the only exception. Like the outcome in G1, most G2 students worked out the functional rules in *Wall Design* using numerical strategies.

Further examination of the data in Table 4.16 uncovered additional striking differences between G1 and G2 students' choices of generalising strategies. The proportion of G1 students using numerical strategies was noticeably higher than that of G2 students in both linear and quadratic tasks. Taking the student performance in *Bricks* as an example, 34 (13%) G1 students used a numerical strategy in rule construction, as compared to 6 (2%) G2 students. This finding suggests that generalising tasks with successive configurations tend to evoke the numerical approach when constructing a functional rule.

Conversely, G2 had generally more students employing the figural strategies successfully in the test than G1, thus highlighting their proclivity for using figural strategies in rule construction. For instance, 35% of G2 students employed a figural strategy to work out a functional rule in *High Chairs* and this figure stood in contrast to 26% of G1 students. Hence, generalising tasks with non-successive configurations appear to encourage the development of the function rule through a figural approach instead of a numerical approach.

Student use of the *guess-and-check* strategy was fairly infrequent in most of the *JuStraGen* tasks, with comparable percentages of students below 10% in both G1 and G2. But the percentages of *indeterminate* (I) strategies were moderately sizeable in certain tasks such as *Bricks*, *Birthday Party Decorations* and *Tulips*. Taking *Bricks* for instance, 17% of G1 students and 12% of G2 students had their strategies classified under this category. When the percentages of G1 and of G2 students in the I category were compared, the values in G2 were found to be higher than those in G1 in all the tasks except for *Bricks* and *Birthday Party Decorations*.

1.3.4 How do the students' generalising strategies vary with the different types of function?

The results in Tables 4.17 and 4.18 reveal some differences in students' use of generalising strategies in linear and in quadratic tasks. The three common types of numerical strategies used in linear tasks were *repeated substitution* (Code 12), *comparison* (Code 13), and *substituting values into formula* (Code 14) whilst the three numerical strategies used in quadratic tasks were *comparison* (Code 13), *finding difference and solving equations* (Code 16), and *grouping* (Code 17).

Of the three numerical strategies observed in linear tasks, the use of the first two strategies was widespread mostly amongst the Express students across all four linear tasks and different formats of pattern display. In total, 86 cases of the use of the repeated substitution strategy and 44 cases of the use of the comparison strategy were observed in this study. As for the quadratic tasks, the *comparison* and *grouping* strategies were spotted in 78 and 27 cases respectively. Although the frequency of the *comparison* strategy was the highest amongst the three numerical strategies used in quadratic tasks, its use was, however, restricted in just one task – *Wall Design*. On the other hand, the grouping strategy found application in three tasks.

For figural strategies, six types were used in linear tasks and eight types in quadratic tasks. Used in 469 cases in linear tasks and 289 cases in quadratic tasks, the large number of cases proved that the *constructive* strategy was by far the students' top favourite. Another popular strategy in both types of tasks was the *reconstructive-constructive* combination (Code 2321), which occurred 42 times in linear tasks and 19 times in quadratic tasks. In spite of the popularity of this *combo* strategy and the *constructive* strategy, the *reconstructive* strategy (Code 23) was not as prevalently used in linear tasks as in quadratic tasks, as the findings had shown. In fact, its use was confined to only one linear task, *Towers*, and its frequency was a low value of two, which was a far cry from the 49 cases in quadratic tasks. In addition to the *reconstructive-constructive* combination, the linear tasks encompassed two other *combo* strategies (Code 2124 and 2324) whereas the quadratic tasks embraced four others (Code 2113, 2123, 2324 and 2423).

4.2.1.6 Summary and discussion

Four categories of generalising strategies had been established and these were *numerical*, *figural*, *guess-and-check*, and *indeterminate*. The strategies used by the Express and Normal (Academic) students were predominantly *figural* but the frequency of *indeterminate* category was also high for the Normal (Academic) students. The numerous cases of students' strategies classified under the *numerical* and *figural* categories suggest that many students were not only familiar with the typical generalising strategies, but were also able to apply them correctly to derive a functional rule. But the fair number of cases in the *guess-and-check* category, together with the considerable number in *indeterminate* categories, also flags a couple of student weaknesses. First, a sizeable number of students might have yet to understand sufficiently some of those common generalising strategies in order to apply them, a similar problem encountered by the students of Moss and Beatty (2006). Next, many students were lacking in clarity as they did not present their thinking clearly for one's ease of comprehension. Lastly, they were also feeble in elaborating their ideas and strategies with sufficient detail.

Five types of numerical strategies had been identified and the two widely used strategies by Express students were *repeated substitution* and *comparison*, both of which have been well studied and extensively discussed in the literature (Bezuszka & Kenney, 2008; Yeo, 2010). However, inferences about the top favourite numerical strategy for Normal (Academic) students could not be drawn from the data due to an extremely small number of students using numerical strategies. The low frequency of *repeated substitution* in the Normal (Academic) course reveals that a significant majority of the less academic students had yet to fully understand this strategy, which is believed to be a clear favourite of Singapore mathematics teachers (Chua & Hoyles, 2010b). Of the remaining three strategies, the literature on pattern generalisation offers relatively little documentation on the use of the *grouping* strategy. Although two cases of application by US students have been described by Rivera and Becker (2011), the strategy was classified as *constructive*. This is why there is hardly any discussion of the *grouping* strategy. But students in the present study used the strategy differently from Rivera and Becker's students.

Nine types of figural strategies had been distinguished, the *constructive* strategy being the most popular amongst the students whilst the *deconstructive* strategy being omitted. These findings concur with a US study by Rivera and Becker (2008) which found that the *deconstructive* strategy was not as well-known and widely used as the *constructive* strategy amongst their students. The *reconstructive* and *figure-ground reversal* strategies were also found to be relatively infrequent, and so were the six *combo* strategies. Although researchers such as Becker and Rivera (2006) had documented previously a type of *combo* strategy which they called *pragmatic* generalisation, and Rivera (2013) had recently reported a student's use of another type of *combo* strategy which he described as transformation-compensation approach, the range of *combo* strategies in the literature on pattern generalisation is by no means as diverse as those observed in the present study.

The higher number of G1 students employing the numerical strategies in both Express and Normal (Academic) courses, coupled with a larger number of Express students in G2 applying the figural strategies, suggests that the format of pattern display may somewhat affect the students' choices of generalising strategies.

4.2.2 STUDENTS' JUSTIFICATION SCHEMES

This section attempts to answer the following main research question:

2. *How do Singapore secondary school students justify the rules they constructed?*

A thorough analysis of the justifications for all correct functional rules was performed to identify the common schemes that students adopted to explain how they developed their rules. After the coding of the student justifications was completed, it was discovered that students had used several kinds of justification schemes and these were then classified by type. Given the establishment of these justification types, a comparison was then carried out to probe the variation of justification types across different student courses, pattern formats and function types. A description of the justification types and the results of the comparison follow next.

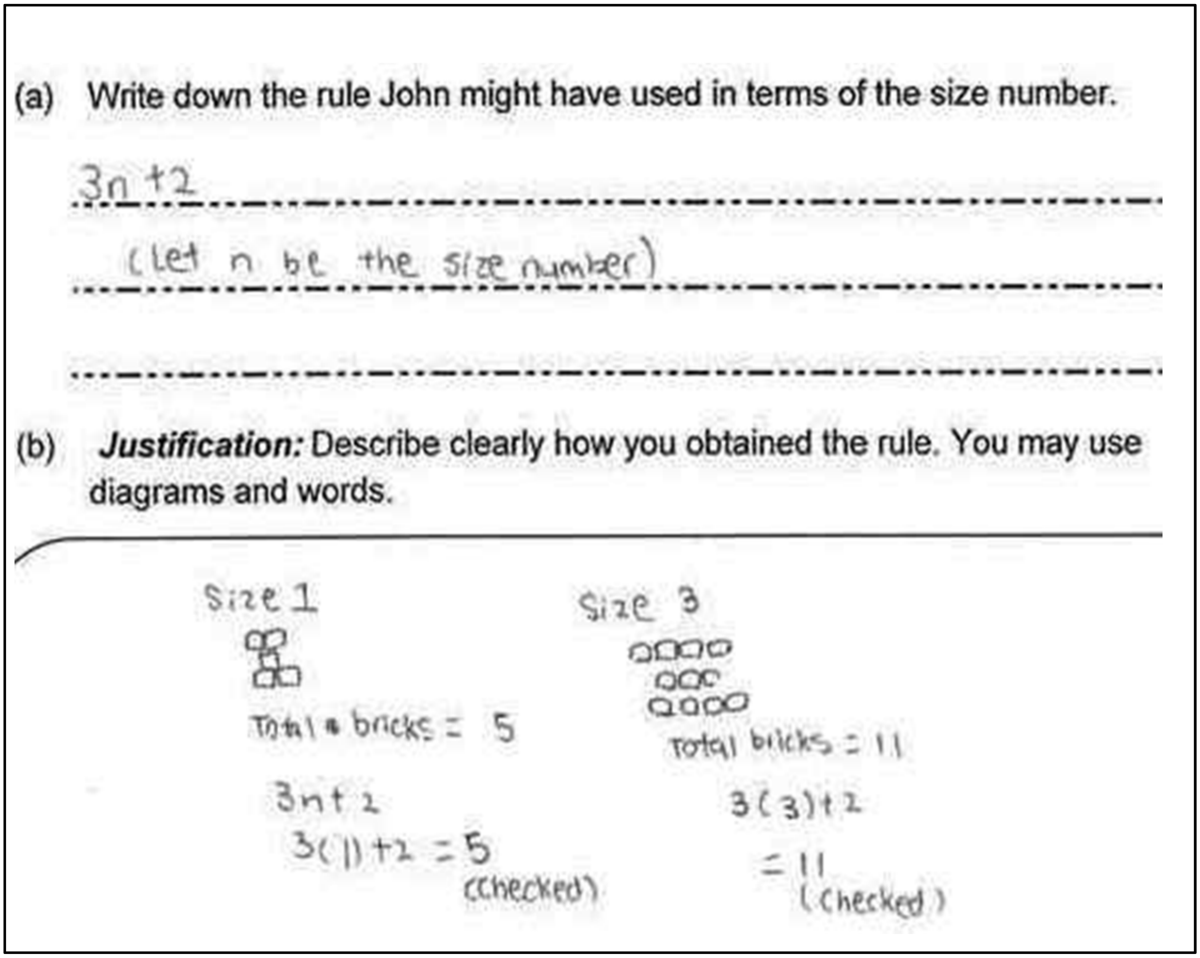
2.1.1 What justification schemes do the students adopt to show how they establish the rule?

The justification schemes for explaining the functional rules were classified into four categories: *justifying recursive rules* (R), *justifying functional rules without diagram* (F), *justifying functional rules with diagrams* (FD) and *miscellaneous* (M). Table 4.19 shows the frequency of justification types used in each linear generalising task by the pattern formats and courses whereas Table 4.20 shows the frequency of justification types used in each quadratic task. As clearly disclosed in both tables, the vast majority of the student justifications belonged chiefly to the F, FD and M categories.

The F category encompasses by far the most number of justification types, with a total of eight as listed below:

- (i) verifying the correctness of the rule by substituting values or by providing diagrams of the configurations (Code 21),
- (ii) organising numerical structures of configurations in a tabular form (Code 22),
- (iii) providing numerical structures of configurations (Code 23),
- (iv) providing a numerical structure of a generic configuration (Code 24),
- (v) describing the steps for obtaining the rule by the comparison strategy (Code 25),
- (vi) describing the steps for obtaining the rule by substituting values into formula (Code 26),
- (vii) providing the working for obtaining the rule by solving equations (Code 28), and
- (viii) demonstrating evidence of guess and check (Code 29).

Examples of the last six justification schemes (codes 23, 24, 25, 26, 28 and 29) were provided previously in Figure 4.12(a), 4.12(d), 4.12(b), 4.11(c), 4.12(c) and 4.3 respectively. Figure 4.15(a) below offers a typical example of the Code–21 justification scheme when the student substituted two values of the size number into the rule to verify that the resulting values match those obtained through counting. An illustration of the Code–22 justification scheme is provided in Figure 4.15(b) when the student tabulated the values of the size number and the corresponding number of bricks, then established a relationship between them.


(a) Write down the rule John might have used in terms of the size number.

3n +2

(let n be the size number)

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Size 1

Total bricks = 5

3n+2

3(1)+2 = 5

(checked)

Size 3

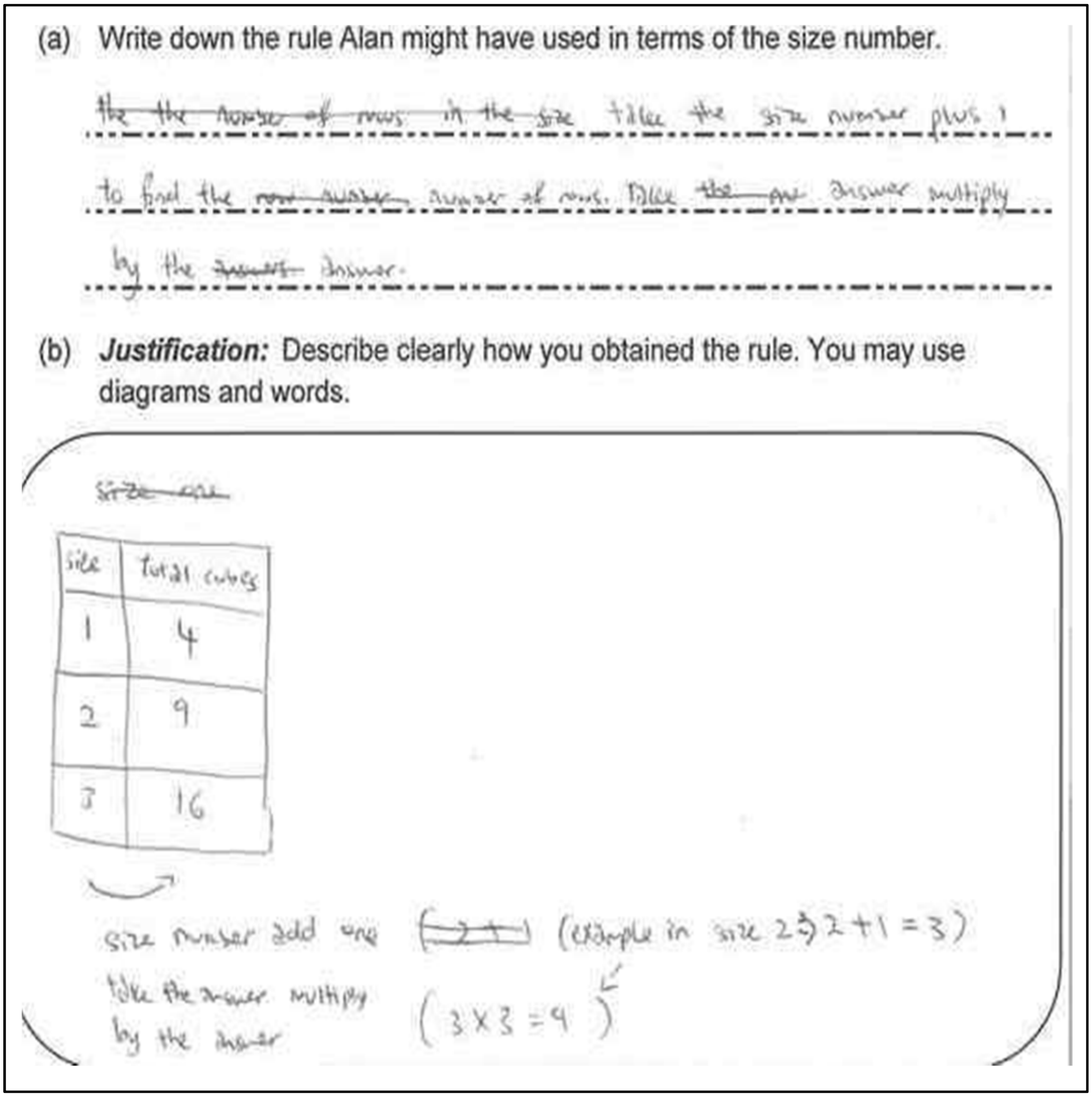
Total bricks = 11

3(3)+2

= 11

(checked)

(a) Code–21 justification scheme



(a) Write down the rule Alan might have used in terms of the size number.

~~the the number of rows in the size~~ take the size number plus 1 to find the ~~row number~~ number of rows. take ~~the~~ ~~one~~ answer multiply by the ~~answer~~ answer.

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

~~size one~~

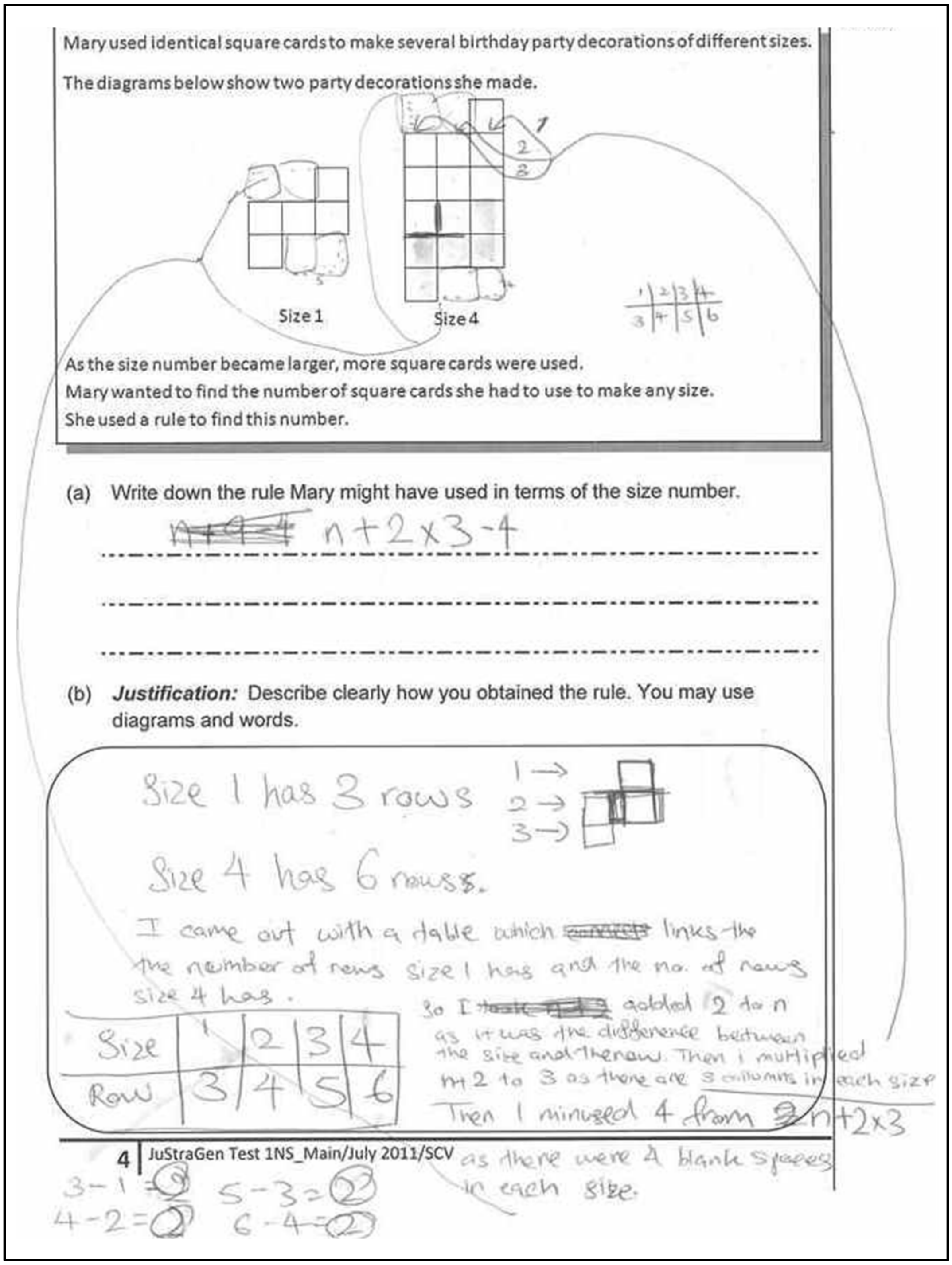
size	total cubes
1	4
2	9
3	16

size number add one ~~(2+1)~~ (example in size 2 → 2 + 1 = 3)

take the answer multiply by the answer (3 × 3 = 9)

(b) Code–22 justification scheme

Figure 4.15. Justification schemes without diagram


Mary used identical square cards to make several birthday party decorations of different sizes.

The diagrams below show two party decorations she made.

Size 1

Size 4

1	2	3	4
3	4	5	6

As the size number became larger, more square cards were used.
Mary wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

(a) Write down the rule Mary might have used in terms of the size number.

~~n+4-4~~ n+2x3-4

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Size 1 has 3 rows 1 → 2 → 3 →

Size 4 has 6 rows.

I came out with a table which ~~connects~~ links the the number of rows size 1 has and the no. of rows size 4 has.

Size	1	2	3	4
Row	3	4	5	6

So I ~~[illegible]~~ added 2 to n as it was the difference between the size and the row. Then I multiplied n+2 to 3 as there are 3 columns in each size. Then I minused 4 from n+2x3 as there were 4 blank spaces in each size.

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3-1=2 5-3=2
4-2=2 6-4=2

Figure 4.16. Justification scheme with diagram and table

The FD category contains three types of justification schemes and these are

(i) providing a few configurations and an elaboration (Code 31),

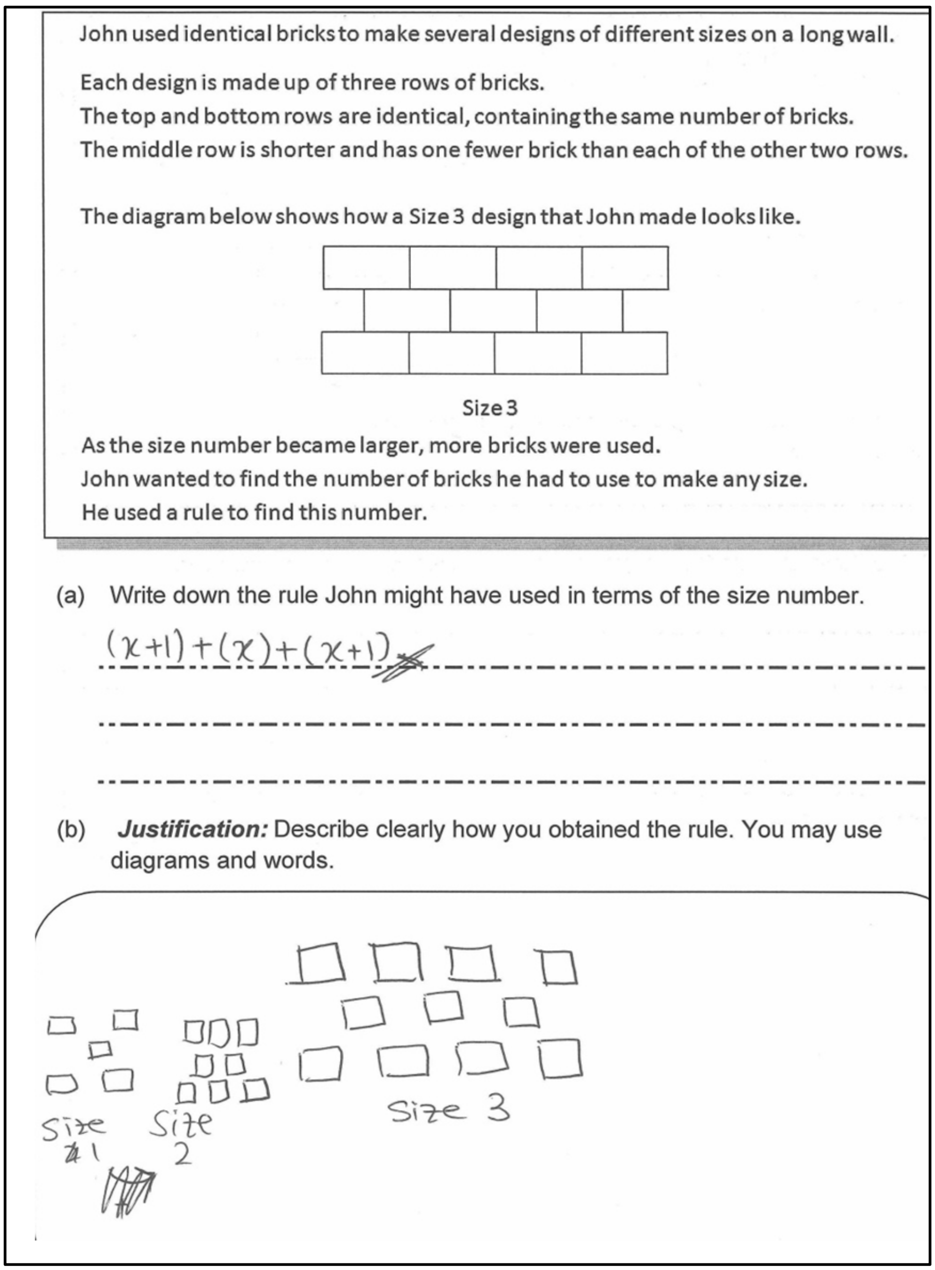
(ii) providing a generic configuration and an elaboration (Code 32), and

(iii) providing a few configurations and organising numerical values in a tabular form (Code 33).

There was a widespread use of the first justification scheme, with 242 cases in linear tasks and 184 cases in quadratic tasks. Equally prevalent was the second justification scheme, spotted 193 times in linear tasks and 161 times in quadratic tasks. As for the third type, there was just one case of use in the entire study. An instance of the Code–31 justification scheme was seen in Figure 4.14(b) when the student drew two configurations to illustrate how the functional rule came about. Figure 4.13(a) showcases an example of Code–32 justification scheme when the student portrayed only a Size–4 configuration to elaborate the development of the functional rule. Figure 4.16 above illustrates the only case of Code–33 justification scheme.

The M category encompasses five justification types, namely, (i) providing configurations only (Code 94), (ii) repeating or restating the rule in another mode (Code 95), (iii) providing a justification related to the rule but failing to elaborate how the rule was obtained (Code 96), (iv) providing a wrong or irrelevant justification (Code 97), and (v) providing no justification (Code 98). The first and fourth types of justification scheme, as the findings of the study had shown, were rather infrequent in the *JuStraGen* test in comparison with a sizeable number of justifications that were coded 96. Examples of these three types of justification schemes are illustrated in Figure 4.17 below.

Figure 4.17(a) shows the student only producing three configurations without any accompanying explanation for the correct functional rule in the *Bricks* task. The *High Chairs* justification in Figure 4.17(b) exemplifies that the correct functional rule is not necessarily derived from following a correct reasoning. The coefficient 3 in the rule $3n + 5$ was actually not the number of cards in the first column as the student had claimed, but rather, it referred to the number of columns in each configuration. As the next two examples in Figures 4.17(c)(i) and 4.17(c)(ii) attest, some students failed to elaborate their thinking and reasoning explicitly enough to illuminate how they obtained the correct functional rules. The working in (c)(i) indicates that although the student found the number of tiles used to build the first four *Towers* configurations and came up with the rule $6 + (n - 1) \times 4$, there was no evidence whatsoever to suggest that the *repeated substitution* strategy, which would normally lead to such a rule, had been applied.



John used identical bricks to make several designs of different sizes on a long wall.

Each design is made up of three rows of bricks.
The top and bottom rows are identical, containing the same number of bricks.
The middle row is shorter and has one fewer brick than each of the other two rows.

The diagram below shows how a Size 3 design that John made looks like.

Size 3

As the size number became larger, more bricks were used.
John wanted to find the number of bricks he had to use to make any size.
He used a rule to find this number.

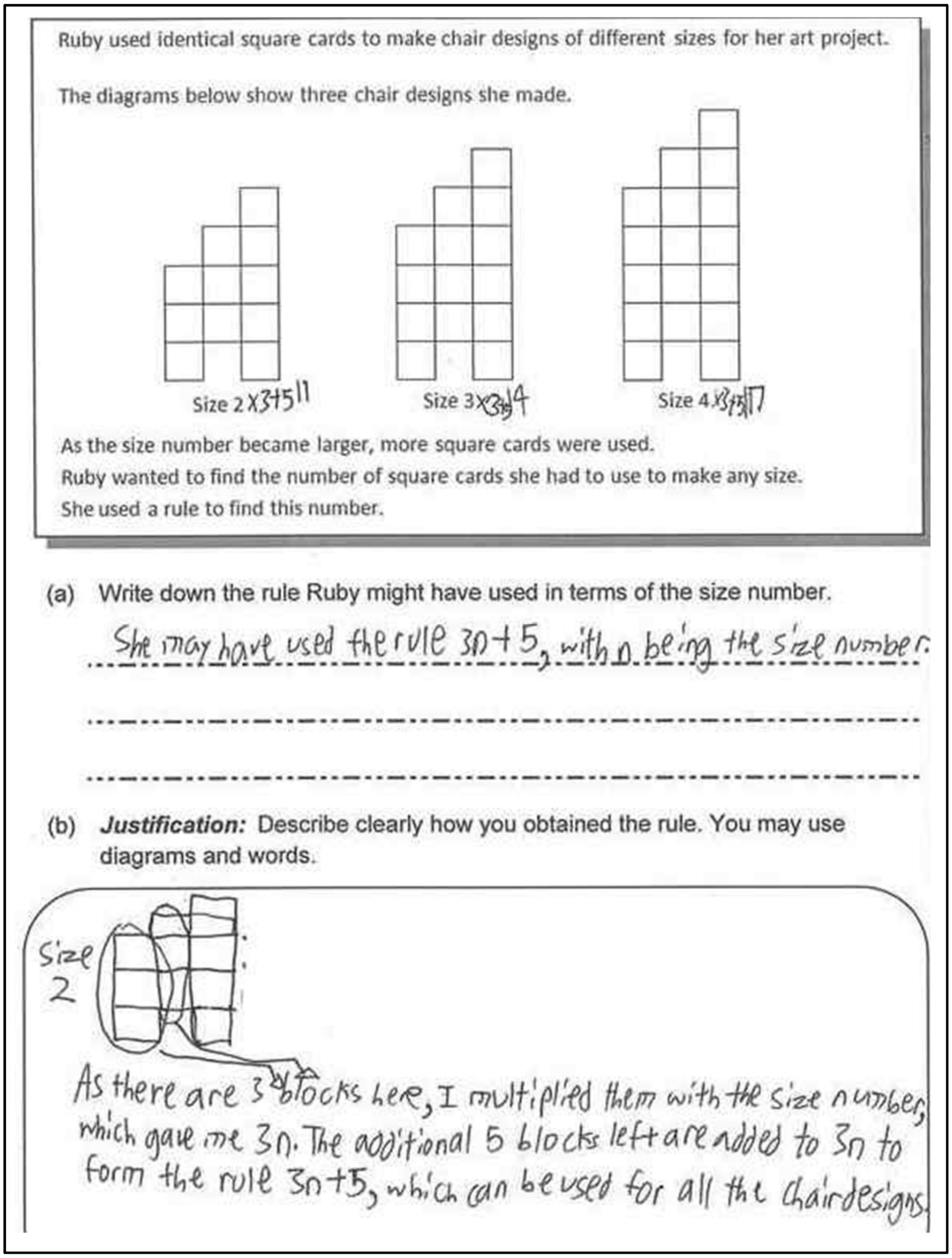
(a) Write down the rule John might have used in terms of the size number.

$(x+1)+(x)+(x+1)$

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Size 1 Size 2 Size 3

(a) Code 94



Ruby used identical square cards to make chair designs of different sizes for her art project.

The diagrams below show three chair designs she made.

Size 2 x3+5 11 Size 3 x3+5 14 Size 4 x3+5 17

As the size number became larger, more square cards were used.
Ruby wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

(a) Write down the rule Ruby might have used in terms of the size number.

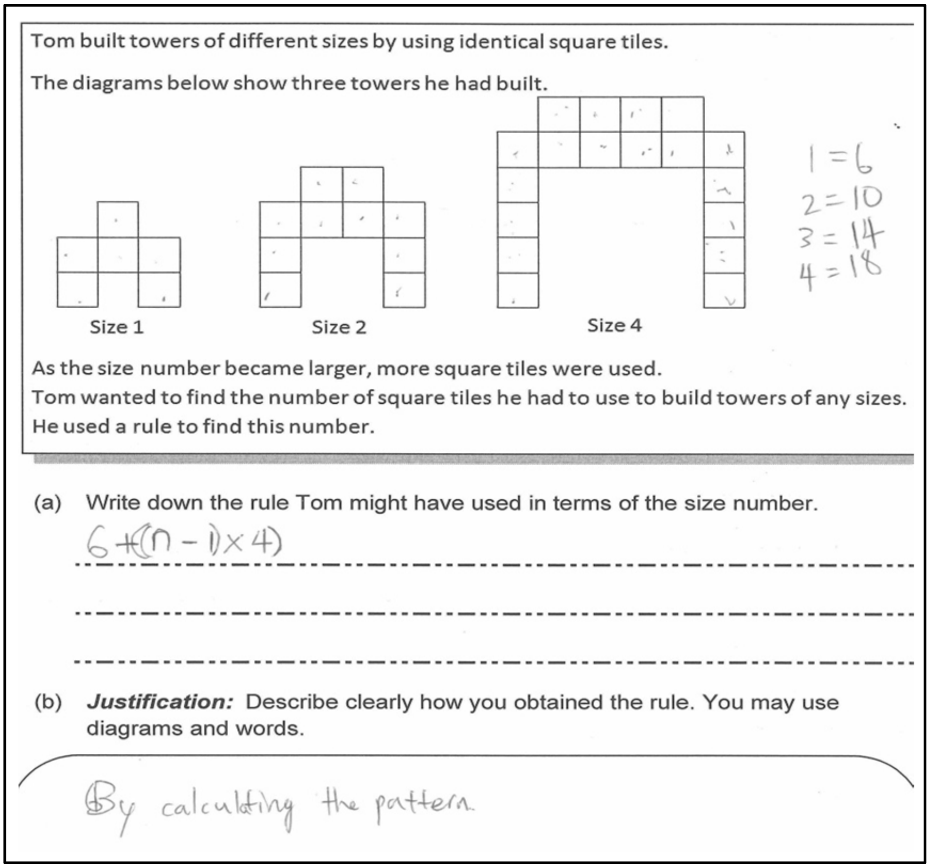
She may have used the rule 3n+5, with n being the size number.

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

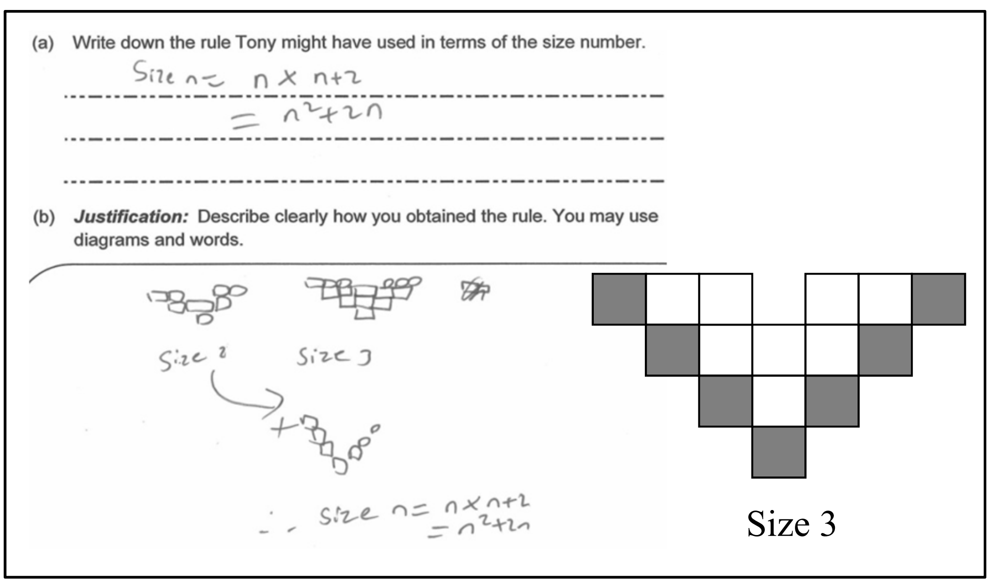
Size 2

As there are 3 blocks here, I multiplied them with the size number, which gave me 3n. The additional 5 blocks left are added to 3n to form the rule 3n+5, which can be used for all the chairdesigns.

(b) Code 97



(c)(i) Code 96

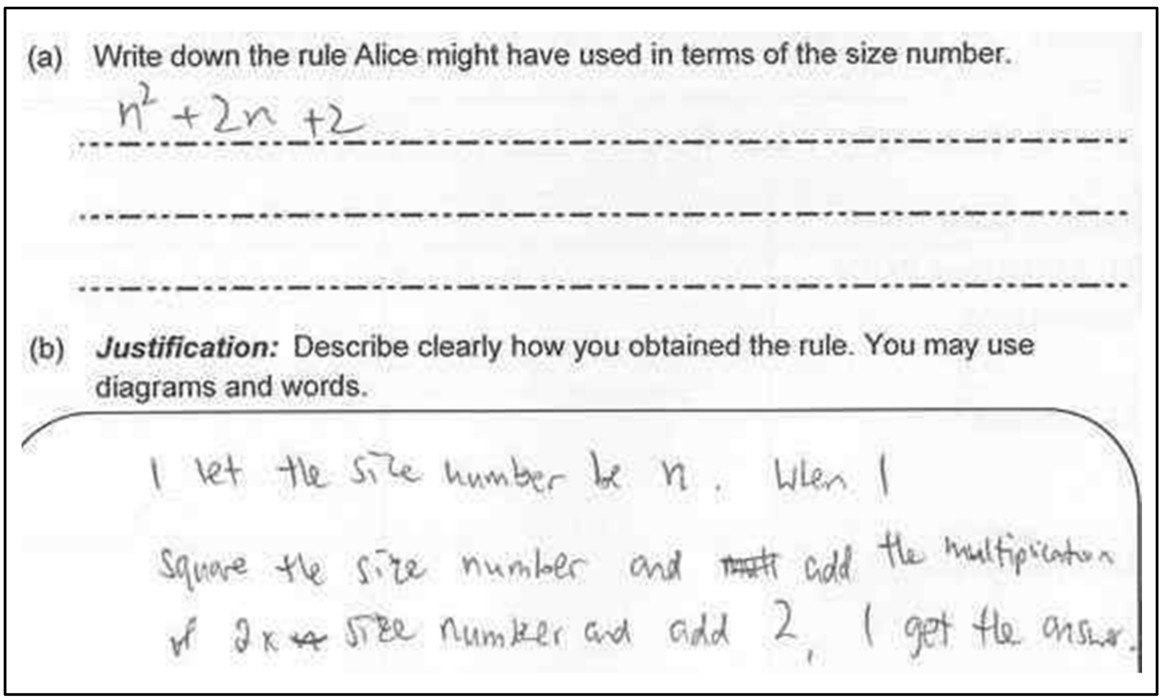


(c)(ii) Code 96

Figure 4.17. Miscellaneous justification schemes

Similarly in (c)(ii), despite illustrating the growth of the *Tulips* configuration from Size 2 to Size 3 by adding a row of tiles arranged in a V shape (refer to the rightmost diagram added by the researcher for a clearer idea of the student's thinking), how the functional rule emerged eventually still remained unexplained (the student's expression of the functional rule, $n \times n + 2$, should have been written correctly as $n \times (n + 2)$).

Apart from the inadequate justifications mentioned above, there were some instances when the students simply repeated the rule or restated it in another mode when asked to justify. This finding shows that those students did not appreciate the justification requirements by clarifying how their rules were obtained. Two examples of such instances are presented in Figure 4.18 below, with (a) showing the student restating the algebraic rule in words and (b) showing another student repeating the rule.

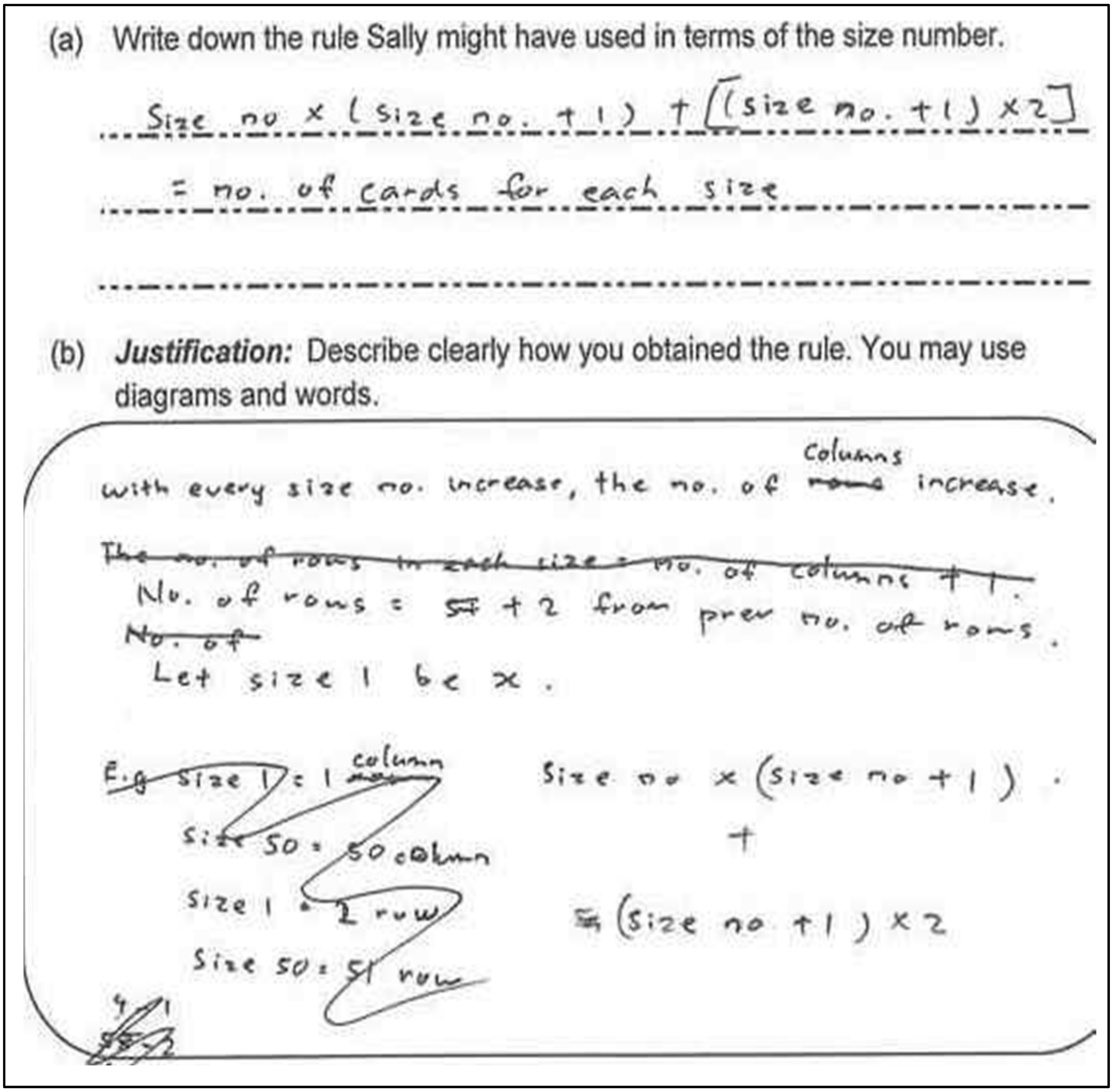

(a) Write down the rule Alice might have used in terms of the size number.

$n^2 + 2n + 2$

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

I let the size number be n. When I square the size number and ~~mult~~ add the multiplication of $2 \times$ ~~4~~ size number and add 2, I get the answer.

(a) Restating the rule



(a) Write down the rule Sally might have used in terms of the size number.

Size no × (size no. + 1) + [(size no. + 1) × 2]

= no. of cards for each size

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

with every size no. increase, the no. of ~~rows~~ columns increase.

~~The no. of rows in each size = no. of columns + 1.~~

No. of rows = ~~5~~ + 2 from prev no. of rows.

~~No. of~~

Let size 1 be x.

E.g size 1 = 1 column

size 50 = 50 column

size 1 = 2 row

size 50 = 51 row

Size no × (size no + 1)

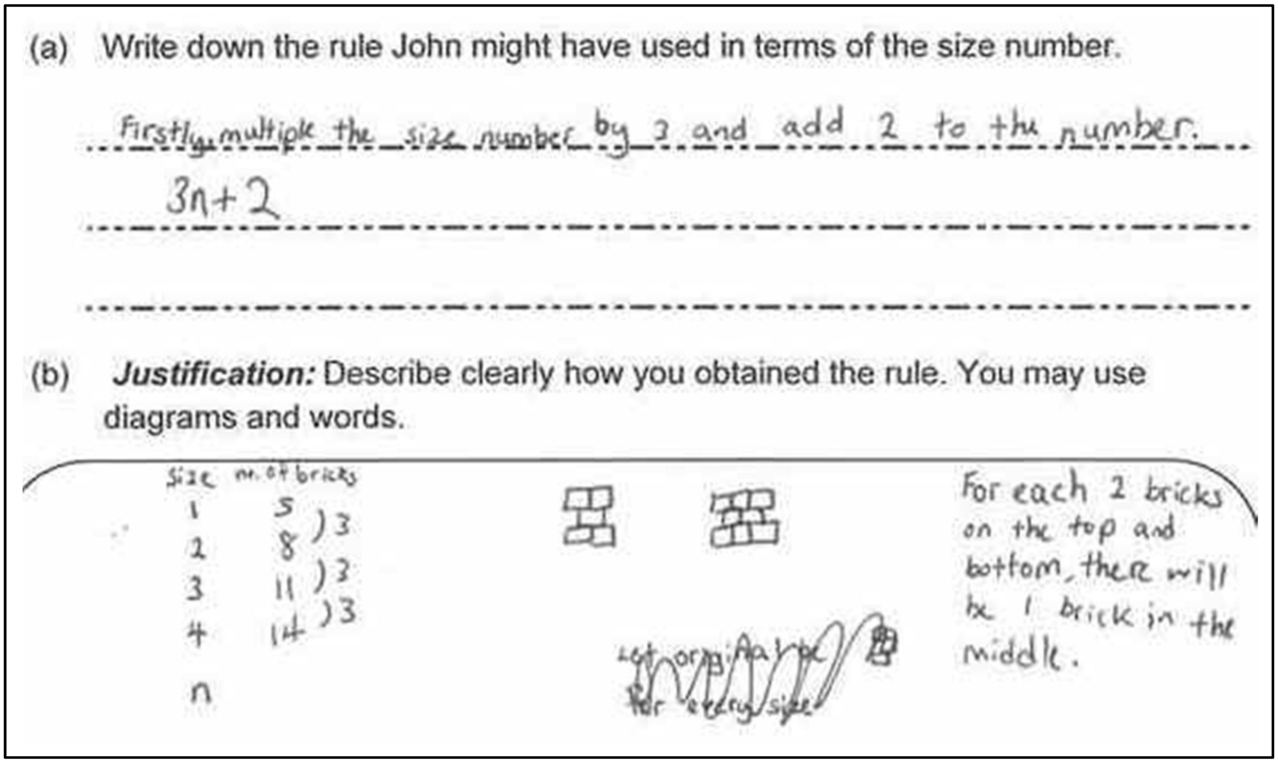
+

(size no + 1) × 2

(b) Repeating the rule

Figure 4.18. Code-95 justification scheme

Finally, a small number of students used justification schemes intended for recursive rules to explain their functional rules. Such schemes were classified under the R category and two types were identified: (i) listing the pattern as a sequence of terms and stating the first difference between terms (Code 11), and (ii) providing a few configurations and stating the first difference between configurations (Code 12). Figure 4.19 below offers an example of each of these two types of justification schemes.


(a) Write down the rule John might have used in terms of the size number.

Firstly, multiple the size number by 3 and add 2 to the number.

$3n+2$

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

size	no. of bricks	
1	5	
2	8	3
3	11	3
4	14	3
n		

For each 2 bricks on the top and bottom, there will be 1 brick in the middle.

(a) Listing the pattern and stating the difference

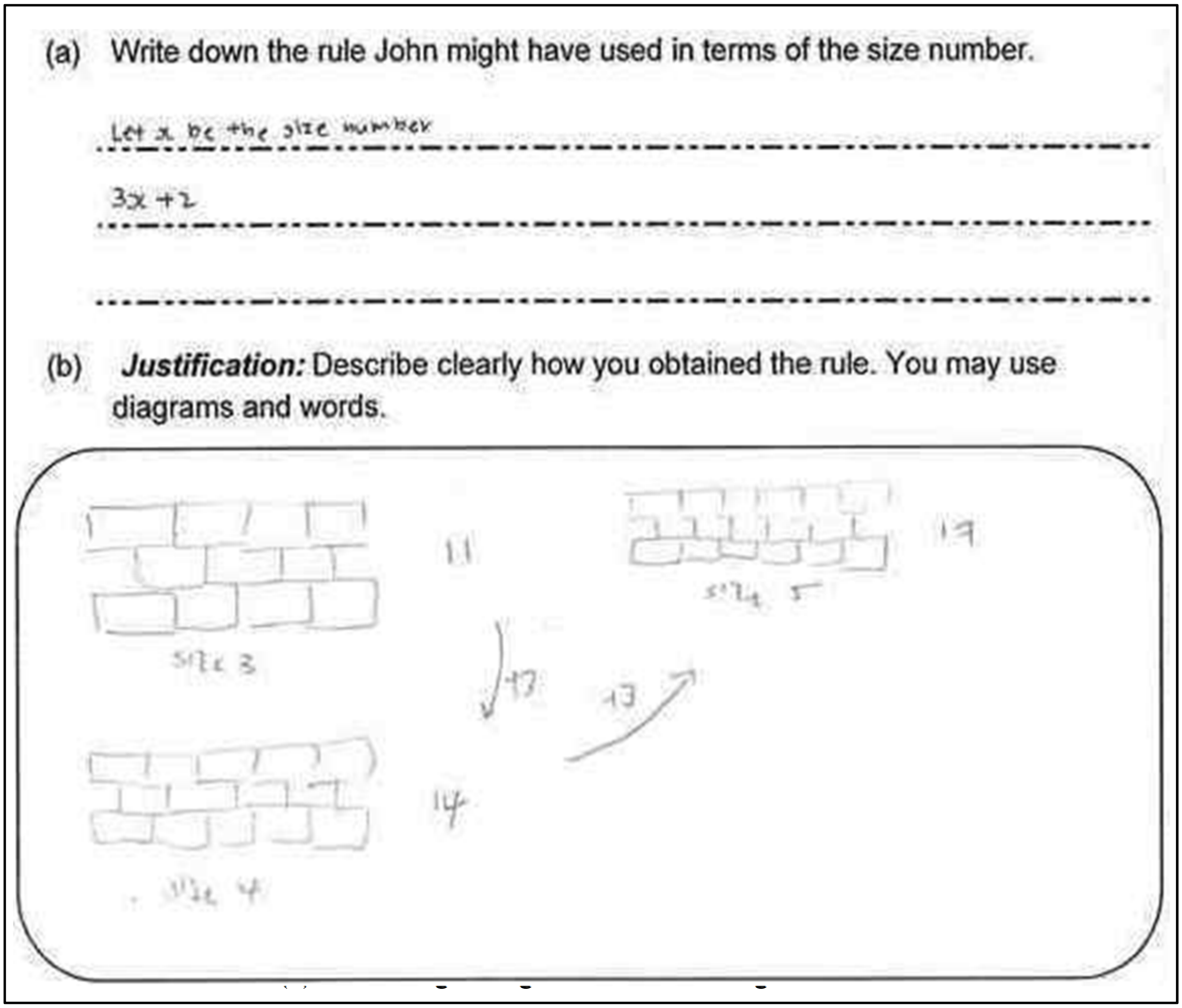


Figure 4.19. Justification schemes intended for recursive rules

2.1.2 How do the students' justification schemes vary with the different courses they are enrolled in?

Overall, as Tables 4.19 and 4.20 below had indicated, both Express and Normal (Academic) students shared a number of common approaches in their justifications. Some of these approaches included *verifying the correctness of the rule* (Code 21), *providing numerical structures of configurations* (Code 23), *providing a numerical structure of a generic configuration* (Code 24), and *providing a few configurations and an elaboration* (Code 31). However, the Express students' justification schemes appeared to concentrate along those coded 21, 23, 24 and 31 whereas those of the Normal (Academic) students centred on mainly the Code–21 and Code–31 types.

On closer examination of the total frequency of each justification type in each *JuStraGen* task, the number of FD justifications amongst the Express students was higher than that of the F justification in five tasks: *Birthday Party Decorations*, *Towers*, *High Chairs*, *Oh Deer*! and *Christmas Party Decorations*. Take *Towers* for instance. Between these two categories, 119 (35%) Express students produced diagrams in their justifications in contrast to 76 (23%) who did not. Similarly in *Christmas Party Decorations*, diagrams were used in 109 (32%) justifications as compared to 55 (16%) justifications without the use of diagrams.

On the other hand, the study seemed to yield mixed results for the Normal (Academic) students. They appeared to favour F over FD justifications when dealing with the linear tasks. Three of the four linear tasks had more F than FD justifications. An example was *Bricks* which comprised 12 F justifications and three FD justifications. But when it came to dealing with quadratic tasks, the results seemed to point to their propensity to switch over to FD justifications. Considering *Oh Deer*! for example, diagrams were seen in seven justifications compared to two without using any diagrams.

Table 4.19: Frequency of types of justification schemes used in linear tasks by pattern formats and courses

		Bricks				Birthday Party Decorations				Towers				High Chairs			
		Express		Normal (Academic)		Express		Normal (Academic)		Express		Normal (Academic)		Express		Normal (Academic)	
Justification	Sample size	170	167	96	82	170	167	96	82	170	167	96	82	170	167	96	82
Type	Schemes	S	NS	S	NS	S	NS	S	NS	S	NS	S	NS	S	NS	S	NS
R	11	1	2			1	1										
	12	1	2	1				1									
	Total	2	4	1		1	1	1									
F	21	31	22	8	1	27	19	6	2	19	22	6		19	24	5	1
	22	4	1			3	1			3				3			
	23	9				11	4			10	4			10	6	1	
	24	10	18	1		11	10	1		4	6			2	7		
	25	8	1		1	9	1			5				6	1		
	26	2	1	1		1		1				1			1		
	28																
	29	1	1				1			2	1	1		1		1	
	Total	65	44	10	2	62	36	8	2	43	33	8		41	39	7	1
FD	31	20	21		1	33	40	3	1	32	33	2	1	22	26	4	3

	32	10	19	1	1	12	21		1	24	30	3	1	25	41	2	2
	33						1										
	Total	30	40	1	2	45	62	3	2	56	63	5	2	47	67	6	5
M	94		1				3				1		1		1	2	
	95	2				1			1	1	3	1	1		3		
	96	11	4	3	1	8	2	3	2	12	12	3	1	14	3	1	
	97			1				1		1	1		1	2			1
	98		2		1		2	1	1	2	4			1	3		
	Total	13	7	4	2	9	7	5	4	16	21	4	4	17	10	3	1

R: justifying recursive rule, F: justifying functional rule without diagram, FD: justifying functional rule with diagrams, M: miscellaneous

S: successive, NS: non-successive; Refer to Appendix 9 for the coding schemes for the justification schemes

Table 4.20: Frequency of types of justification schemes used in quadratic tasks by pattern formats and courses

		Wall Design				Christmas Party Decorations				Oh Deer!				Tulips			
Justification		Express		Normal (Academic)		Express		Normal (Academic)		Express		Normal (Academic)		Express		Normal (Academic)	
Scheme	Sample size	170	167	96	82	170	167	96	82	170	167	96	82	170	167	96	82
Type	Scheme	S	NS	S	NS	S	NS	S	NS	S	NS	S	NS	S	NS	S	NS
R	11										1				1		
	12														1		
	Total										1				2		
F	21	20	21	3		11	15			18	18		1	22	22	3	1
	22	4						2		3	1			1	1		
	23	15	10			11	3			9	4			19	19	1	
	24	2	1			6	6			9	8			4	6		
	25	5	2	1													
	26																
	28	1				1				1	1			1			
	29	1	2			1	1			3	1	1		3	2	1	
	Total	48	36	4		30	25	2		43	33	1	1	50	50	5	1
FD	31	21	10	1	1	35	25	6	3	22	17	2	3	19	19		

	32	11	21	1	1	13	36	1	1	23	29	2		7	14	1	
	33																
	Total	32	31	2	2	48	61	7	4	45	46	4	3	26	33	1	
M	94						1	1			1	1					
	95	1	3				5		1	1	1	1		1	1		
	96	7	9	1	1	7	6	1	1	4	4	1	1	10	5	1	1
	97					4	1								1	1	
	98		4			1	3				2			1	1		
	Total	8	16	1	1	12	16	2	2	5	8	3	1	12	8	2	1

R: justifying recursive rule, F: justifying functional rule without diagram, FD: justifying functional rule with diagrams, M: miscellaneous

S: successive, NS: non-successive; Refer to Appendix 9 for the coding schemes for the justification schemes

2.1.3 How do the students' justification schemes vary with the different formats of pattern display?

Table 4.21 below tabulates the frequencies and percentages of the four categories of justification schemes by pattern formats. Considering the outcomes of G1 students, the category with the highest percentage of students in each *JuStraGen* task was either F or FD. *Bricks*, *Birthday Party Decorations*, *Wall Design* and *Tulips* were the four tasks with the most number of students justifying their rules without using any diagrams and the rest of the tasks had the most number of students using diagrams in their justifications. The outcomes in G2 were similar and there were now more tasks with most number of students in the FD category. *Birthday Party Decorations*, *Towers*, *High Chairs*, *Oh Deer*! and *Christmas Party Decorations* were the five tasks having the most number of students justifying their rules with diagrams whilst the number of justifications without any diagrams peaked in the remaining three tasks.

Another observation emerging from the careful analysis of the justification schemes in Table 4.21 was that there were apparently more G1 than G2 students choosing to justify without drawing any diagrams. The absolute difference in percentage point between G1 and G2 students who created such justifications varied from as high as 11% in *Birthday Party Decorations* (with 26% of G1 students and 15% of G2 students) to a low of 1% in *Tulips* (with 21% of G1 students and 20% of G2 students). On the contrary, G2 boasted more students using diagrams in their justifications than G1. In *High Chairs*, the number of students who justified their rules using diagrams soared from 53 (20%) in G1 to 72 (29%) in G2. These findings indicate that students' justification schemes might have been influenced by the format of pattern display that they worked with.

In general, the use of miscellaneous (M) justification schemes was relatively uncommon in most *JuStraGen* tasks when compared to the F and FD categories. The percentages of M across different formats of pattern display were comparable in each task.

Table 4.21: Frequency and percentage of justification types by pattern formats

	Successive (G1) (n = 266)								Non-successive (G2) (n = 249)							
	R		F		FD		M		R		F		FD		M	
Tasks	freq	%	freq	%	freq	%	freq	%	freq	%	freq	%	freq	%	freq	%
Bricks	3	1	75	28	31	12	17	6	4	2	46	18	42	17	9	4
Birthday Party Decorations	2	1	70	26	48	18	14	5	1	1	38	15	64	26	11	4
Towers			51	19	61	23	20	8			33	13	65	26	25	10
High Chairs			48	18	53	20	20	8			40	16	72	29	11	4
Wall Design			52	20	34	13	9	3			36	14	33	13	17	7
Christmas Party Decorations			32	12	55	21	14	5			25	10	65	26	18	7
Oh Deer!			44	17	49	18	8	3	1	1	34	14	49	20	9	4
Tulips			55	21	27	10	14	5	2	1	51	20	33	13	9	4

R: justifying recursive rule, F: justifying functional rule without diagram, FD: justifying functional rule with diagrams, M: miscellaneous

2.1.4 How do the students' justification schemes vary with the different types of function?

The results in Tables 4.19 and 4.20 reveal some commonalities in students' choice of justification schemes for linear and quadratic tasks. Nearly all the listed justification schemes were observed in both types of tasks. Of the eight justification schemes not involving diagrams, *verifying the correctness of the rule* (Code 21) was by far the most pervasive scheme used in both types of tasks, with 212 instances in linear tasks and 155 instances in quadratic tasks. The next two prevalent schemes in each type of tasks were the same and these were *providing numerical structures of configurations* (Code 23) and *providing a numerical structure of a generic configuration* (Code 24). However, there were more cases of the latter than the former in linear tasks, and vice-versa in quadratic tasks.

For justification schemes involving diagrams, the students' top two preferences for both linear and quadratic tasks were *providing a few configurations and an elaboration* (Code 31) and *providing a generic configuration and an elaboration* (Code 32). The ratio of Code–31 type to Code–32 type was about 5 : 4 in both types of tasks.

Although the miscellaneous category encompassed five justification schemes, only the use of *providing a justification related to the rule but failing to elaborate how the rule was obtained* (Code 96) was fairly frequent in both linear and quadratic tasks. Students had applied this scheme in a total of 80 cases under linear tasks as well as a total of 60 cases under quadratic tasks.

4.2.2.1 Summary and discussion

Four categories of justification schemes had been established and these were *justifying functional rules without diagram* (F), *justifying functional rules with diagrams* (FD), *justifying recursive rules* (R), and *miscellaneous* (M). A vast number of student justifications were classified under F and FD, particularly in the Express course. Eight types of justification schemes were found under F, with the empirical verification of the validity of a rule being the most common amongst all the students. This suggests that whilst many students elaborated their thinking and reasoning clearly, some had not realised that verification is not compelling enough to be an argument to convince others of the truth of the rule. Crucially, it does not explain how the rule is formulated. Three types of justification schemes were found under FD, with providing a generic or a few configurations being the most prevalent. In addition, the manifestation of justifications under R and M illustrates that although students can often construct the correct functional rule, their development of an adequate and convincing justification is not always guaranteed.

Many Express students justified their rules with diagrams, which matches the high frequency of figural strategies. On the other hand, the Normal (Academic) students justified linear rules without diagrams and quadratic rules with diagrams, which were consistent with the frequencies of numerical and figural strategies. However, their frequency of M was quite high in some tasks such as *Bricks*, *Birthday Party Decorations*, *Oh Deer*! and *Tulips*. No differences in the use of justification schemes were detected between linear and quadratic tasks, thus showing the approaches were applicable across all types of figural generalising tasks.

4.3 EFFECT OF TASK FEATURES

This section attempts to answer the following main research question:

3 *How do task features influence Singapore secondary school students' rule construction?*

The present study probed systematically the possible effect of two task features on students' rule construction and these task features were the format of pattern display and the type of functions. The following two null hypotheses were postulated:

$H1_0$ The format of pattern display has no significant effect on the students' rule construction.

$H2_0$ The type of functions has no significant effect on the students' rule construction.

The corresponding alternative hypotheses were stated as follows:

$H1_1$ The format of pattern display produces a significant difference on the students' rule construction.

$H2_1$ The type of functions produces a significant difference on the students' rule construction.

To investigate the effect of the format of pattern display, an independent-measures design was conducted on two groups of students, G1 and G2. G1 students dealt with successive configurations whereas G2 students were given non-successive configurations. After the students' responses in the *JuStraGen* test were scored, an independent-samples *t* test was performed to evaluate the first null hypothesis $H1_0$.

A repeated-measures design was conducted for each group of students to study whether their generalisations were successful when the rule changed from linear to quadratic. A paired-samples *t* test was then performed to evaluate the second null hypothesis $H2_0$.

The outputs of the different *t* tests are reported in the rest of this section when answering the sub-research questions.

3.1.1 Is there any effect of the format of pattern display on the Express students' rule construction?

Table 4.22 summarises the descriptive statistics for Express students on their performance in linear tasks, quadratic tasks and the entire test. For instance, the mean score of G1 students for all four linear tasks was 29.5 with a standard deviation of 10.4, and the mean score of G2 students for the entire *JuStraGen* test was 51.7 with a standard deviation of 22.6.

Table 4.22: Mean scores and standard deviations of linear and quadratic tasks for Express students by pattern format

		n	mean	sd
Linear tasks	Successive (G1)	170	29.5	10.4
	Non-successive (G2)	167	28.6	11.2
Quadratic tasks	Successive (G1)	170	24.0	12.1
	Non-successive (G2)	167	23.1	12.6
Entire test	Successive (G1)	170	53.5	20.8
	Non-successive (G2)	167	51.7	22.6

Table 4.23 below displays the corresponding independent *t* test statistics. Express students in G1 performed better in linear tasks (m = 29.5, sd = 10.4) than their counterparts in G2 (m = 28.6, sd = 11.2). However, this difference was not statistically significant, $t(335) = .778$, $p > .05$. Similarly, G1 students (m = 24.0, sd = 12.1) also outperformed G2 students (m = 23.1, sd = 12.6) in quadratic tasks and, again, this difference was not statistically significant, $t(335) = .663$, $p > .05$. Overall, the mean score of G1 students for the entire *JuStraGen* test (m = 53.5, sd = 20.8) was not statistically significantly different ($t(335) = .765$, $p > .05$) from that of G2 students (m = 51.7, sd = 22.6). To sum up, the null hypothesis $H1_0$ was accepted at the .05 level of significance for the case of Express students.

Table 4.23: Independent t test output for Express students

	t	df	Sig (2-tailed)
Linear tasks	.778	335	.437
Quadratic tasks	.663	335	.508
Entire test	.765	335	.445

3.1.2 Is there any effect of the format of pattern display on the Normal (Academic) students' rule construction?

The descriptive statistics for Normal (Academic) students on their performance in linear tasks, quadratic tasks and the entire test are summarised in Table 4.24.

Table 4.24: Mean scores and standard deviations of linear and quadratic tasks for Normal students by pattern format

		n	mean	sd
Linear tasks	Successive (G1)	96	16.8	8.4
	Non-successive (G2)	82	12.1	7.2
Quadratic tasks	Successive (G1)	96	9.5	6.8
	Non-successive (G2)	82	6.2	6.4
Entire test	Successive (G1)	96	26.3	13.3
	Non-successive (G2)	82	18.4	12.0

For instance, the mean score of G2 students for linear tasks was 12.1 with a standard deviation of 7.2, and the mean score of G1 students for the entire *JuStraGen* test was 26.3 with a standard deviation of 13.3.

Table 4.25 below summarises the corresponding independent *t* test statistics. Normal (Academic) students in G1 (m = 16.8, sd = 8.4) outperformed their G2 counterparts (m = 12.1, sd = 7.2) in linear tasks and there was a statistically significant difference in their means, $t(176) = 4.0$, $p < .05$. The value of Cohen's d was 0.60, which indicated a medium effect size. The mean scores for quadratic tasks between G1 (mean = 9.5, sd = 6.8) and G2 (m = 6.2, sd = 6.4) students also differed significantly, $t(176) = 3.3$, $p < .05$ and the effect size, d, was 0.50, a medium value. The mean score of G1 students for the entire *JuStraGen* test (m = 26.3, sd = 13.3) was significantly higher than that of G2 students (m = 18.4, sd = 12.0), $t(176) = 4.2$, $p < .05$. The value of d was 0.63, indicating a medium effect size. In summary, the null hypothesis $H1_0$ was rejected at the .05 level of significance and the conclusion was that Normal (Academic) students found generalising tasks with successive configurations significantly easier than those with non-successive configurations.

Table 4.25: Independent t test output for Normal (Academic) students

	t	df	Sig (2-tailed)	Effect size, *d*
Linear tasks	4.0*	176	.000	0.60
Quadratic tasks	3.3*	176	.001	0.50
Entire test	4.2*	176	.000	0.63

* significant at .05 level

3.2.1 Is there any effect of the type of function on the Express students' rule construction?

Table 4.26 provides the descriptive statistics for the 337 Express students on their performance in all *JuStraGen* tasks, organised in matching pairs. For instance, *Bricks* is a linear task whose matching quadratic task is *Wall Design*. The students' mean score of

Bricks was 6.9 with a standard deviation of 3.4, whilst their mean score of *Wall Design* was 5.7 with a standard deviation of 4.1. In all the four pairs of generalising tasks, the Express students, on the average, fared better in linear tasks than in quadratic tasks.

Table 4.26: Mean scores and standard deviations of JuStraGen tasks for Express students

			Paired Differences	
	mean	sd	mean	sd
Bricks	6.9	3.4	1.2	4.2
Wall Design	5.7	4.1		
Birthday Party Decorations	7.3	3.3	1.2	3.6
Christmas Party Decorations	6.1	4.2		
Towers	7.5	3.2	1.4	3.6
Oh Deer!	6.1	4.0		
High Chairs	7.3	3.5	1.6	3.9
Tulips	5.7	3.7		

Table 4.27 below summarises the corresponding paired *t* test statistics. The score of *Wall Design* differed from that of *Bricks* by an average of m = 1.2 with a standard deviation of 4.2. This difference was statistically significant, $t(336) = 5.2$, $p < .000$. The value of Cohen's d was 0.29, which indicated a medium effect size. For the *Birthday Party Decorations – Christmas Party Decorations* pair, the difference between the two mean scores is 1.2 with a standard deviation of 3.6. This difference was also statistically significant, $t(336) = 6.2$, $p < .000$ and the effect size, d, was 0.33, a medium value. The mean scores between *Towers* and *Oh Deer*! differed significantly as well ($t(336) = 7.4$, $p < .000$) with the mean score of *Towers* being significantly higher than that of *Oh Deer*!. The mean difference was 1.4 with a standard deviation of 3.6. The value of d was 0.40, denoting a medium effect size. Finally for the last pair of tasks, the mean score of *High Chairs* was statistically higher than that of *Tulips*, $t(336) = 7.6$, $p < .000$, with a medium effect size of 0.42. The mean difference was 1.6 with a standard deviation of 3.9. To summarise the

findings, the null hypothesis $H2_0$ was rejected at the .05 level of significance and the conclusion was very clear: Express students did find quadratic generalising tasks significantly harder to do than linear generalising tasks.

Table 4.27: Paired t test output for Express students

	t	df	Sig (2-tailed)	Effect size, *d*
Bricks – Wall Design	5.2*	336	.000	0.29
Birthday Party Decorations – Christmas Party Decorations	6.2*	336	.000	0.33
Towers – Oh Deer!	7.4*	336	.000	0.40
High Chairs - Tulips	7.6*	336	.000	0.42

* significant at .05 level

3.2.2 Is there any effect of the type of function on the Normal (Academic) students' rule construction?

Table 4.28 provides the descriptive statistics for the 178 Normal (Academic) students on their performance in all *JuStraGen* tasks, organised in matching pairs. The students' mean score of *Towers* was 3.6 with a standard deviation of 2.6, whilst their mean score of *Oh Deer!* was 1.9 with a standard deviation of 2.6. The mean difference for this pair of generalising tasks was 1.7 with a standard deviation of 3.5. Like their counterparts in the Express course, the Normal (Academic) students, on the average, also performed better in linear tasks than in quadratic tasks in all four pairs.

Table 4.28: Mean scores and standard deviations of JuStraGen tasks for Normal (Academic) students

			Paired Differences	
	mean	sd	mean	sd
Bricks	3.5	2.7	1.4	3.3
Wall Design	2.1	2.6		
Birthday Party Decorations	3.7	2.6	1.6	3.1
Christmas Party Decorations	2.1	2.8		

Towers	3.6	2.6	1.7	3.5
Oh Deer!	1.9	2.6		
High Chairs	3.8	2.7	1.9	3.4
Tulips	1.9	2.3		

The corresponding paired *t* test statistics are displayed in Table 4.29 below. The mean score of *Bricks* was statistically significantly higher than that of *Wall Design*, $t(177) = 5.8$, $p < .000$, with a medium effect size of 0.44. The difference between the mean scores of *Birthday Party Decorations* and *Christmas Party Decorations* was also statistically significant, $t(177) = 7.1$, $p < .000$. The effect size, d, was 0.54, a medium value. In much the same way, the mean scores between *Towers* and *Oh Deer*! differed significantly as well ($t(177) = 6.5$, $p < .000$) with the mean score of *Towers* being significantly higher than that of *Oh Deer*!. The value of d was 0.49, indicating a medium effect size. Finally, the mean score of *High Chairs* was also significantly higher than that of *Tulips*, $t(177) = 7.4$, $p < .000$, with a medium effect size of 0.55. In conclusion, the outcome of the paired *t* test has clearly failed to accept the null hypothesis $H2_0$ at the .05 level of significance and there was evidence pointing to the fact that Normal (Academic) students did find quadratic generalising tasks significantly harder to accomplish than linear generalising tasks.

Table 4.29: Paired t test output for Normal (Academic) students

	t	df	Sig (2-tailed)	Effect size, *d*
Bricks – Wall Design	5.8*	177	.000	0.44
Birthday Party Decorations – Christmas Party Decorations	7.1*	177	.000	0.54
Towers – Oh Deer!	6.5*	177	.000	0.49
High Chairs - Tulips	7.4*	177	.000	0.55

* significant at .05 level

4.3.1 Summary and discussion

Generalising tasks involving non-successive configurations had been found to be significantly harder than those with successive configurations for the Normal (Academic) students, but for the Express students, no significant differences were observed between the two pattern formats in linear and quadratic tasks. These results manifest that more able students can still construct a correct functional rule even if the pattern deviates from the typical and familiar format of three successive configurations to one involving non-successive configurations.

Students' ability in rule construction seems to be assisted very much by their awareness of the structure inherent in the pattern. To become aware of the structure, some students in the present study needed to draw additional configurations themselves before they could see the structural relationship from the geometrical arrangement of tiles or cards. For some other students, drawing such configurations was not necessary at all. By treating the given configurations generically, they were able to abstract the structural relationship from them. For instance, some students allocated the non-successive format discerned Size 1 of *Birthday Party Decorations* as a row of three cards plus two more, Size 4 as four rows of three cards plus two more, and hence, Size *n* as *n* rows of three cards plus two more, or $3n+2$ when expressed in symbols (see, for instance, Figure 4.11(a)). This finding lends support to the view of Mason, Stephens and Watson (2009) and Küchemann (2010) that teaching students to identify structure in the learning of mathematics is crucial. This is because being able to recognise structure is an extremely useful skill for students to acquire in that their attention will no longer be drawn to focus on the usual counting of tiles or cards but on abstracting relationships between sets of objects, then followed by articulating a rule that captures this relationship.

The quadratic generalising tasks had also been found to be significantly more testing than the linear generalising tasks for both Express and Normal (Academic) students. So there is now vital evidence to believe strongly that the types of functions underpinning a pattern can contribute to student difficulties in making generalisations. With this finding, student performance in quadratic tasks reported in earlier studies can now be explained.

4.4 SUMMARY OF STUDY I

In response to the three main research questions in Study I stated in Chapter 2, the findings of Study I have led to the following six key conclusions about the participating students' generalisations and justifications, as well as the effect of the format of pattern display and the type of functions on their generalisations and justifications.

First, the success of a sizeable number of Express students in linear *JuStraGen* tasks suggests that the fundamental concepts of pattern generalisation, ranging from the idea of examining the relationship between the position number and its corresponding term and searching for an invariant relationship between them to abstracting this relationship as a functional rule, were generally well understood and the students were fairly adept in rule construction. However, their modest success rates in quadratic tasks also suggest that although the students performed rather competently in routine and familiar linear tasks, some of them faltered when working with the less familiar quadratic tasks in the *JuStraGen* test. This implies that the unsuccessful students were not able to build on their experiences with linear tasks to deal with the quadratic tasks.

Second, a vast majority of Normal (Academic) students did not manage to answer the basic linear tasks correctly, let alone the quadratic tasks. The low success rates in both types of generalising tasks indicate clearly that there was a widespread failure on their part to understand fully the basic generalisation concepts. The prevalence of recursive rules particularly in the linear tasks not only contrasted greatly to a substantial number of functional rules formulated by their Express counterparts, but also provided further evidence of the students' lack of appreciation of the task requirement for a functional rule.

Third, Express students established a far more diverse range of structurally distinct-looking but equivalent functional rules for both the linear and quadratic tasks than the Normal (Academic) students. The rules were predominantly expressed in algebraic notations.

Fourth, most Express students engaged a wide range of numerical and figural generalising strategies to develop their rules, with a few strategies probably new to the research literature. But a sizeable number of cases of *guess-and-check* and *indeterminate* strategies also reveals that some students were not only unfamiliar with the common generalising strategies but also unable to articulate their ideas and strategies clearly.

Fifth, the Express students tended to use diagrams in their justifications of the rules whereas the Normal (Academic) students justified linear rules without diagrams and quadratic rules with diagrams.

Finally, the format of pattern display produced a significant difference on only the Normal (Academic) students' rule construction whereas the type of functions had a significant effect on both the Express and Normal (Academic) students' rule construction.

The next chapter will report the results of Study II, an exploration of the kind of generalising strategy that the students judged as the most helpful for rule construction, as well as the efficacy of their choice of best-help strategy on rule construction.

CHAPTER 5 : RESULTS OF STUDY II

A major finding from Study I is that Singapore secondary school students envisioned the patterns in the *JuStraGen* test in several ways and established numerous equivalent functional rules through using a variety of generalising strategies. With so many different strategies available, it is unclear which one would best help them to construct a functional rule successfully. Study II was designed to examine (a) their beliefs of what they would judge as the most helpful generalising strategy for establishing linear and quadratic functional rules, and (b) the efficacy of their choice of best-help strategy on their generalisation. The data used in this investigation were collected from the same sample of 515 students who took the *JuStraGen* test in Study I through administering the *QBBS* questionnaire which comprised the four generalising tasks: *Birthday Party Decoration*s, *Bricks*, *High Chairs* and *Christmas Party Decoration*s. The first two tasks, both linear, were administered immediately after the first *JuStraGen* test, whilst the remaining two, one linear and one quadratic, were given after the second *JuStraGen* test. Given that the data about student beliefs were collected through a survey, it was important to explore in greater depth the validity of the data collected in order to offer greater confidence in the findings. 16 students comprising eight from each course were therefore selected for individual interviews to determine whether they could derive the functional rules correctly using their chosen generalising strategies.

This chapter reports the findings related to students' choices of best-help generalising strategies from the survey responses, and their attempts to develop a functional rule using the chosen generalising strategy. It begins with an overview of the survey findings for the *QBBS* tasks, then follows with a discussion of any significant difference between the Express students' choices of best-help strategies and those of the Normal (Academic) students. It ends with a description of how the students' *QBBS* choices compared with their generalising strategies in the *JuStraGen* test, as well as how helpful their *QBBS* choices appeared to be in the formulation of rules.

5.1 OVERVIEW OF RESULTS OF STUDY II

This section begins with a synopsis of the students' preferences of best-help strategies in the four *QBBS* tasks. Overall, the *constructive* strategy was especially popular amongst all the students, being the top favourite approach of not only the Express students in *Bricks*, *High Chairs* and *Christmas Party Decorations*, but also the Normal (Academic) students in all four tasks. However, in *Birthday Party Decorations*, the *reconstructive* strategy emerged the top choice for the Express students, followed closely behind by the *constructive* strategy. The least popular approach in *Birthday Party Decorations* and *Bricks* was the *deconstructive* strategy whereas it was the *figure-ground reversal* strategy in *High Chairs* and *Christmas Party Decorations*.

Within the group of students assigned to the successive pattern format, the distribution of Express students' choices of best-help strategies in the QBBS task was not statistically different from that of the Normal (Academic) students' in three of the four tasks. There was, however, significant difference in three tasks between the distribution for Express students and the distribution for Normal (Academic) students working with non-successive pattern format.

When the choices of beat-help strategies made by the entire Express student sample were studied and then compared, no significant differences were revealed between the distribution of choices for those dealing with patterns in successive format and the distribution for those working with non-successive format in all the four *QBBS* tasks. A comparison of the choices made by all the Normal (Academic) students also did not detect any significant differences between the distribution for successive format and the distribution for non-successive format in three tasks.

Students from both courses were interviewed to assess whether or not they were able to establish a functional rule using the strategies they judged as most helpful. Four fifths of them were successful in constructing their rules, eight succeeding without any guidance. Five students derived their rules after some guidance whereas the remaining three still failed despite the guidance.

5.2 RESULTS OF QBBS SURVEY

This section seeks to answer the fourth and last main research question as stated below.

4 *What do Singapore secondary school students judge to be the most helpful generalising strategy for constructing the functional rule?*This research question comprises eight sub-questions which will be addressed in the subsequent four sections.

5.2.1 STUDENTS' CHOICES OF BEST-HELP GENERALISING STRATEGIES

This section focusses on the first six sub-questions concerning the types of generalising approach in each *QBBS* task that Express and Normal (Academic) students believed would help them to develop the functional rule. Each approach depicted a different generalising strategy. An examination of the Express students' choices of strategies precedes that of the Normal (Academic) students' choices.

4.1.1 What generalising strategies do the Express students believe would best help them to generate the linear functional rule?

Table 5.1 below shows the frequency and percentage of Express students in each of the four different approaches of working out the rules presented in the three linear *QBBS* tasks by the format of pattern display. All the four different approaches were selected by students. Broadly speaking, the *constructive* strategy appeared to be the students' clear favourite, considering it being their top choice of best-help strategy in two of the three tasks: *Bricks* and *High Chairs*. This was then followed in descending order by the *reconstructive*, *repeated substitution* and, finally, *deconstructive* strategies.

In *Bricks*, the *constructive* strategy was selected by 37% of the Express students assigned to deal with successive configurations, 48% of the Express students working with non-successive configurations, and 43% of the Express students overall. The corresponding percentages in *High Chairs* were 32%, 38% and 35% respectively. The *reconstructive* and *repeated substitution* strategies were picked by 33% and 22% of the Express students respectively. The *deconstructive* approach, preferred by only 8 (2%) Express students, was the least popular choice of generalising strategies in *Bricks*. In *High Chairs*, 111 (33%) students selected *reconstructive*, 69 (20%) selected *repeated substitution*, and 39 (12%)

chose *figure-ground reversal*, making the last approach the most unpopular of the four strategies.

In *Birthday Party Decorations*, although the *reconstructive* strategy topped the list of four strategies with 34% of the Express students, the *constructive* strategy followed fairly close behind with 31% of the Express students, missing the top spot narrowly by 3% of the Express students. Further examination of the data revealed that the *reconstructive* strategy was favoured by 32% of those working with successive pattern format and 36% of those dealing with non-successive pattern format whereas the corresponding percentages for the *constructive* strategy were 32% and 31% respectively. Following these top two strategies were the *repeated substitution* (28%) and *deconstructive* (7%) strategies in descending order of percentage.

Table 5.1: Frequency and percentage of Express students' choices of strategies for linear tasks

Strategies (Strategy code)	Birthday Party Decorations			Bricks			High Chairs		
	S (n = 170)	NS (n = 167)	Total (n = 337)	S (n = 170)	NS (n = 167)	Total (n = 337)	S (n = 170)	NS (n = 167)	Total (n = 337)
Repeated substitution (Code 12)	54	41	95	44	31	75	37	32	69
	(32%)	(25%)	(28%)	(26%)	(19%)	(22%)	(22%)	(19%)	(20%)
Constructive (Code 21)	54	51	105	62	81	143	54	64	118
(i.e., breaking whole into parts)	(32%)	(31%)	(31%)	(37%)	(48%)	(43%)	(32%)	(38%)	(35%)
Reconstructive (Code 23)	54	61	115	60	51	111	57	54	111
(i.e., rearranging parts)	(32%)	(36%)	(34%)	(35%)	(31%)	(33%)	(33%)	(33%)	(33%)
Deconstructive (Code 22)	8	14	22	4	4	8	NA	NA	NA
(i.e., overlapping of parts)	(4%)	(8%)	(7%)	(2%)	(2%)	(2%)			
Figure-ground reversal (Code 24)	NA	NA	NA	NA	NA	NA	22	17	39
(i.e., viewing configuration as part of a large structure)							(13%)	(10%)	(12%)

S: successive, NS: non-successive, NA: Not available in the task

4.1.2 What generalising strategies do the Express students believe would best help them to generate the quadratic functional rule?

Table 5.2 below shows the frequency and percentage of Express students in each of the four different approaches presented in the only quadratic *QBBS* task by the format of pattern display.

Table 5.2: Frequency and percentage of Express students' choices of strategies for quadratic task

	Christmas Party Decorations		
Strategies (Strategy code)	S (n = 170)	NS (n = 167)	Total (n = 337)
Repeated substitution (Code 12)	28 (16%)	28 (17%)	56 (17%)
Constructive (Code 21)	74 (44%)	61 (36%)	135 (40%)
(i.e., breaking whole into parts)			
Reconstructive (Code 23)	41 (24%)	55 (33%)	96 (28%)
(i.e., rearranging parts)			
Figure-ground reversal (Code 24)	27 (16%)	23 (14%)	50 (15%)
(i.e., viewing configuration as part of a large structure)			

S: successive, NS: non-successive

The *constructive* strategy was picked by 74 (44%) Express students presented with successive configurations and 61 (36%) Express students presented with non-successive configurations, totaling 135 (40%) Express students. Mirroring the survey outcomes in *High Chairs*, the next two favoured strategies were *reconstructive* (28%), then followed by *repeated substitution* (17%), with the percentage of the former higher than that of the latter by 11% of the Express students. The most unpopular strategy was *figure-ground reversal* (15%), chosen by fewer than a fifth of the Express students.

4.2.1 What generalising strategies do the Normal (Academic) students believe would best help them to generate the linear functional rule?

Table 5.3 tabulates the frequency and percentage of Normal (Academic) students in the various different approaches presented in the three linear *QBBS* tasks by the format of pattern display.

Table 5.3: Frequency and percentage of Normal (Academic) students' choices of strategies for linear tasks

	Birthday Party Decorations			Bricks			High Chairs		
Strategies (Strategy code)	S (n = 96)	NS (n = 82)	Total (n = 178)	S (n = 96)	NS (n = 81)	Total (n = 177)	S (n = 95)	NS (n = 82)	Total (n = 177)
Repeated substitution (Code 12)	29	27	56	30	20	50	18	28	46
	(30%)	(33%)	(31%)	(31%)	(24%)	(28%)	(19%)	(34%)	(26%)
Constructive (Code 21)	43	40	83	41	45	86	27	27	54
(i.e., breaking whole into parts)	(45%)	(49%)	(47%)	(43%)	(55%)	(48%)	(28%)	(33%)	(30%)
Reconstructive (Code 23)	15	9	24	20	10	30	35	12	47
(i.e., rearranging parts)	(16%)	(11%)	(14%)	(21%)	(12%)	(17%)	(36%)	(15%)	(26%)
Deconstructive (Code 22)	9	6	15	5	6	11	NA	NA	NA
(i.e., overlapping of parts)	(9%)	(7%)	(8%)	(5%)	(7%)	(6%)			
Figure-ground reversal (Code 24)	NA	NA	NA	NA	NA	NA	15	15	30
(i.e., viewing configuration as part of a large structure)							(16%)	(18%)	(17%)

S: successive, NS: non-successive, NA: not applicable

The students' choice that topped the list of best-help strategies consistently in all the three generalising tasks was the *constructive* approach, with nearly half of the Normal (Academic) student sample selecting it in *Birthday Party Decorations* (47%) and *Bricks* (48%), and a third of them in *High Chairs* (30%). Those who were assigned the non-successive pattern format appeared to be particularly fond of this strategy, with as high as 49% and 55% in *Birthday Party Decorations* and *Bricks* respectively. Subsequently, following in descending order were *repeated substitution* and *reconstructive*. Taking *Birthday Party Decorations* for instance, there were over twice as many Normal (Academic) students picking *repeated substitution* as *reconstructive*. The least preferred strategy in *Birthday Party Decorations* and *Bricks* was *deconstructive*, both tasks comprising fewer than a tenth of the Normal (Academic) students, and in *High Chairs*, it was *figure-ground reversal*, chosen by slightly fewer than a fifth of the Normal (Academic) students.

4.2.2 What generalising strategies do the Normal (Academic) students believe would best help them to generate the quadratic functional rule?

Table 5.4 organises the frequency and percentage of Normal (Academic) students in the various different approaches presented in the quadratic *QBBS* tasks according to the format of pattern display. The survey outcomes of *Christmas Party Decorations* paralleled very much those of the linear tasks. An overall of 64 (36%) Normal (Academic) students preferred clearly the *constructive* strategy to the other three strategies over which the remaining students were evenly distributed. Although the *figure-ground reversal* strategy was a second favourite (26%) of those who dealt with the successive pattern format ahead of *reconstructive* (20%) and *repeated substitution* (19%), it was found to be the least popular amongst those working with the non-successive pattern format, with a percentage of 16% behind 26% for *repeated substitution* and 22% for *reconstructive*.

Table 5.4: Frequency and percentage of Normal (Academic) students' choices of strategies for quadratic task

	Christmas Party Decorations		
Strategies (Strategy code)	S (n = 96)	NS (n = 82)	Total (n = 178)
Repeated substitution (Code 12)	18 (19%)	21 (26%)	39 (22%)
Constructive (Code 21)	34 (35%)	30 (37%)	64 (36%)
(i.e., breaking whole into parts)			
Reconstructive (Code 23)	19 (20%)	18 (22%)	37 (21%)
(i.e., rearranging parts)			
Figure-ground reversal (Code 24)	25 (26%)	13 (16%)	38 (21%)
(i.e., viewing configuration as part of a large structure)			

S: successive, NS: non-successive

4.3.1 Is there any difference in the distribution of students' choices of best-help generalising strategies between the Express and Normal (Academic) students?

This sub-question examined students' choices of the most helpful generalising strategies and probed whether there was a relationship between their choices and the courses they were enrolled in. The null and alternative hypotheses were postulated as follows:

$H3_0$ The distribution of best-help generalising strategies is the same for Express and Normal (Academic) students.

$H3_1$ The distribution of best-help generalising strategies for Express students is different from the distribution for Normal (Academic) students.

In total, eight χ^2-tests were performed to evaluate the null hypothesis $H3_0$, one for each *QBBS* task between the Express and Normal (Academic) students working with each format of pattern display. Table 5.5 provides the descriptive statistics for these eight χ^2-tests conducted.

Table 5.5: χ^2-test statistics between Express and Normal (Academic) students for each format of pattern display

	Birthday Party Decorations		Bricks		High Chairs		Christmas Party Decorations	
	χ^2 (df = 3)	p	χ^2 (df = 3)	p	χ^2 (df = 3)	p	χ^2 (df = 3)	p
Between Express and Normal (Academic) students in G1	11.16	.011*	7.00	.072	1.00	.801	4.93	.177
Between Express and Normal (Academic) students in G2	19.27	.000*	12.27	.007*	14.89	.002*	4.61	.202

*Significant at $p < .05$

As Table 5.5 indicates, the χ^2-test demonstrated that the distribution of best-help generalising strategies between the Express and Normal (Academic) students working with the successive format of pattern display was not statistically different in *Bricks*, *High Chairs* and *Christmas Party Decorations*. A significant difference was present between the distribution of best-help strategies for Express students and the distribution for Normal (Academic) students working with successive format in *Birthday Party Decorations* (χ^2 = 11.16, df = 3, p < .05). The value of *Cramer's V* was .21, indicating a medium effect size according to Cohen's (1998) conventions. For a better understanding of this remarkable finding in *Birthday Party Decorations*, the observed and expected frequencies of best-help strategies for this task according to the student courses, provided in Table 5.6 below, were examined further.

Table 5.6: Observed and expected frequencies of best-help strategies for Birthday Party Decorations presented in successive pattern format

	Observed data				Expected data			
	Repeated substitution	Constructive	Reconstructive	Deconstructive	Repeated substitution	Constructive	Reconstructive	Deconstructive
Express (n = 170)	54	54	54	8	53	62	44	11
Normal (Academic) (n = 96)	29	43	15	9	30	35	25	6

Clearly from the above table, there were more Express students preferring the *reconstructive* strategy than would be expected and fewer of them favouring the *constructive* strategy than would be expected. Conversely, more Normal (Academic) students preferred the *constructive* strategy than would be expected and fewer of them favoured the *reconstructive* strategy than would be expected.

For students working with the non-successive pattern format, there were significant differences between the distribution of best-help strategies for Express students and the distribution for Normal (Academic) in all three linear tasks, namely, *Birthday Party Decorations* (χ^2 = 19.27, df = 3, p < .05), *Bricks* (χ^2 = 12.27, df = 3, p < .05), and *High Chairs* (χ^2 = 14.89, df = 3, p < .05). The respective *Cramer's V* values were .28, .22, and .24, all indicating medium effect sizes. *Christmas Party Decorations* is the only exception where the difference in distribution of best-help generalising strategies between Express and Normal (Academic) students was not statistically significant. Table 5.7 to Table 5.9

below provide the observed and expected frequencies of best-help strategies for *Birthday Party Decorations*, *Bricks*, and *High Chairs* respectively, and these data were examined to describe the details of the significant results.

Table 5.7: Observed and expected frequencies of best-help strategies for Birthday Party Decorations presented in non-successive pattern format

	Observed data				Expected data			
	Repeated substitution	Constructive	Reconstructive	Deconstructive	Repeated substitution	Constructive	Reconstructive	Deconstructive
Express (n = 167)	41	51	61	14	46	61	47	13
Normal (Academic) (n = 82)	27	40	9	6	22	30	23	7

Table 5.8: Observed and expected frequencies of best-help strategies for Bricks presented in non-successive pattern format

	Observed data				Expected data			
	Repeated substitution	Constructive	Reconstructive	Deconstructive	Repeated substitution	Constructive	Reconstructive	Deconstructive
Express (n = 167)	31	81	51	4	34	85	41	7
Normal (Academic) (n = 81)	20	45	10	6	17	41	20	3

Table 5.9: Observed and expected frequencies of best-help strategies for High Chairs presented in non-successive pattern format

	Observed data				Expected data			
	Repeated substitution	Constructive	Reconstructive	Figure-ground reversal	Repeated substitution	Constructive	Reconstructive	Figure-ground reversal
Express (n = 167)	32	64	54	17	40	61	44	22
Normal (Academic) (n = 82)	28	27	12	15	20	30	21	11

The three tables reveal that the preference for the *reconstructive* strategy soared higher than would be expected amongst the Express students assigned to non-successive pattern format but dipped for the Normal (Academic) students working with the same pattern format. The fluctuation between the observed and expected frequencies was the greatest in *Birthday Party Decorations*. For the Normal (Academic) students, there was a greater preference for the *constructive* strategy in *Birthday Party Decorations*, and for the *repeated substitution* strategy in *High Chairs*.

4.3.2 Is there any difference in the distribution of students' choices of best-help generalising strategies between the successive and non-successive format of pattern display?

This sub-question examined whether there was a relationship between the students' choices of best-help generalising strategies and the pattern format that they worked with. The null and alternative hypotheses were postulated as follows:

$H4_0$ The distribution of best-help generalising strategies is the same for successive and non-successive formats of pattern display.

$H4_1$ The distribution of best-help generalising strategies for successive format of pattern display is different from the distribution for non-successive format of pattern display.

Like in the previous sub-question, a total of eight χ^2-tests were also performed to evaluate the null hypothesis $H4_0$, one for each *QBBS* task between successive and non-successive formats of pattern display in each student course. Table 5.10 provides the descriptive statistics for these eight χ^2-tests conducted.

Table 5.10: χ^2-test statistics between successive and non-successive pattern formats in each student course

	Birthday Party Decorations		Bricks		High Chairs		Christmas Party Decorations	
	χ^2 (df = 3)	p	χ^2 (df = 3)	p	χ^2 (df = 3)	p	χ^2 (df = 3)	p
Between S and NS formats in Express course	3.90	.272	5.48	.140	1.91	.592	3.59	.310
Between S and NS formats in Normal (Academic) course	1.19	.756	4.37	.224	12.54	.006*	3.22	.360

*Significant at $p < .05$

As Table 5.10 indicates, the χ^2-test demonstrated that the distribution of best-help generalising strategies between the successive and non-successive formats of pattern display was not statistically different in any of the *QBBS* tasks in the Express course. Similar findings were found in the Normal (Academic) course, with *High Chairs* as the only exception. In that task, there was a significant difference between the distribution of best-help strategies for successive format and the distribution for non-successive format ($\chi^2 = 12.54$, df = 3, $p < .05$), with a *Cramer's V* value of .27, indicating a medium effect size. To understand better why only *High Chairs* displayed a significant difference, its observed and expected frequencies of best-help strategies according to the student courses are provided in Table 5.11 below for further examination.

Table 5.11: Observed and expected frequencies of best-help strategies for High Chairs by pattern format

	Observed data				Expected data			
	Repeated substitution	Constructive	Reconstructive	Figure-ground reversal	Repeated substitution	Constructive	Reconstructive	Figure-ground reversal
Successive (n = 95)	18	27	35	15	25	29	25	16
Non-successive (n = 82)	28	27	12	15	21	25	22	14

Clearly from the above table, more Normal (Academic) students assigned to the successive format selected the *reconstructive* strategy than would be expected and fewer of them opted for *repeated substitution* than would be expected. Conversely, more of those assigned to the non-successive format picked *repeated substitution* and fewer selected *reconstructive* than would be expected.

5.2.1.1 Summary and discussion

The Express students' top choice of best-help strategy for the linear tasks varied between the *constructive* and *reconstructive* strategies whereas the *constructive* strategy was their clear favourite for the quadratic task. However, the highest frequencies of students selecting the former strategy in two linear (*Bricks* and *High Chairs*) and one quadratic (*Christmas Party Decorations*) generalising tasks, together with the relatively large number of students choosing that strategy in *Birthday Party Decorations*, support favourably the idea that the Express students demonstrated a stronger preference for the *constructive* strategy over the *reconstructive* strategy. This finding suggests clearly that a substantial majority of the Express students preferred to work out the functional rule using a figural approach.

Another striking result emerging from the Express data set is that the *repeated substitution* strategy was not as highly preferred as the *constructive* and *reconstructive* strategies. This finding contrasts with results of a previous study by Chua and Hoyles (2010b) with a different sample, that is of 16 secondary school mathematics teachers who were asked to work out individually a functional rule for *Birthday Party Decorations* using the strategy that they would employ in their classroom demonstration. The majority engaged a numerical strategy, including *repeated substitution*, and the rest used the *constructive* strategy. Thus, although teachers might often assume that *repeated substitution* would appeal to their students, the present study offers some evidence to show that students may hold differing views from the teachers and in particular, for example, many Express students definitely do not find *repeated substitution* helpful.

For Normal (Academic) students, their choices of best-help strategy did not seem to vary widely across the different *QBBS* tasks. The *constructive* strategy, having received the highest frequency in every task, was by far their clear favourite, and this was subsequently followed by the *repeated substitution* strategy. The second popular strategy for the Normal (Academic) students interestingly contrasted with the top two choices for the Express students. A reason for the prevalence of *repeated substitution* amongst the Normal (Academic) students might be that such a strategy was not only commonly featured in the local mathematics textbook, but also a widely taught method in school. Thus this strategy might be the most familiar to students (Chua & Hoyles, 2010b). However, its infrequent use in the *JuStraGen* test suggests that students may have yet to understand the strategy in a depth sufficient to apply it to develop a correct functional rule. This strategy, although it shows clearly how the number of cards or tiles used to create each configuration changes with the size number in an orderly tabular format, is not as straightforward as it might appear for deriving the quadratic rule. The sizeable frequency of the *repeated substitution* strategy in *Christmas Party Decorations* in the survey reveals that many students had underestimated the complexity of deriving the quadratic rule using such a strategy and believed naively that it would be helpful in rule construction.

The *deconstructive* strategy was highly unpopular in *Birthday Party Decorations* and *Bricks* for both the Express and Normal (Academic) students, so was the *figure-ground reversal* strategy in *High Chairs* and *Christmas Party Decorations*. These findings are in complete agreement with the earlier findings about the students' choice of generalising strategies in the *JuStraGen* test.

The findings show that there were no differences in student choice of best-help strategy between Express and Normal (Academic) students working with successive format in all but one linear task. However, there were differences between Express and Normal (Academic) students working with non-successive format in all the linear tasks. Additionally, there were also no differences in the choice of best-help strategy between the two pattern formats in the Express course and in the Normal (Academic) course, with only an exception in the latter.

5.2.2 STUDENTS' CHOICES OF BEST-HELP GENERALISING STRATEGIES VERSUS THEIR STRATEGIES IN JUSTRAGEN TEST

This section compares the generalising strategies that students used in the *JuStraGen* test with their choices of best-help strategies in the *QBBS* survey in an attempt to answer the following sub-research question.

4.4 How do the students' choice of best-help generalising strategies compare with their generalising strategies used in the *JuStraGen* test?

Table 5.12 tabulates the frequency of the Express students' generalising strategies in the *JuStraGen* test against their *QBBS* choices for *Birthday Party Decorations* and *Bricks*, and Table 5.13 provides the frequency for *High Chairs* and *Christmas Party Decorations*. It is clear from both tables that amongst the users of numerical strategies in the test, the frequency of *repeated substitution* in the survey was the highest of all four given approaches in each *QBBS* task, except for *Bricks* wherein the frequency of *repeated substitution* was lower than the frequency of *reconstructive* by just 1% of numerical strategy users. Of those who used *repeated substitution* in the test, nearly half of them judged their strategy as the most helpful. The same strategy was also popular amongst those who produced a recursive rule instead of a functional rule. However, taking the frequencies of the three figural approaches (i.e., *constructive*, *reconstructive*, and *deconstructive* or *figure-ground reversal*) in each *QBBS* task collectively, the majority of the users of numerical strategies did not believe *repeated substitution* was the most helpful strategy for deriving the functional rule and the percentages of such users varied from 53% in *High Chairs* and 58% each in *Birthday Party Decorations* and *Christmas Party Decorations* to 65% in *Bricks*.

Users of figural strategies in the test strongly believed that either the *constructive* or the *reconstructive* strategy would best help them to formulate the functional rules, with at least

80% of them in both approaches combined in each *QBBS* task. With the only exception in *Birthday Party Decorations*, the majority of those who employed the *constructive* strategy in the test perceived such a strategy as most helpful in generating the functional rules in *Bricks*, *High Chairs* and *Christmas Party Decorations*. However, a relatively small number of *constructive* strategy users switched to believing in *repeated substitution* in these three tasks. In *Birthday Party Decorations*, the *reconstructive* strategy emerged the students' top choice of best-help strategies. Interestingly, although the *reconstructive* strategy was not frequently engaged in the test in three of the four generalising tasks, the data analysis reveals that nearly a third of the Express students judged it as a best-help strategy. Similarly, the *deconstructive* strategy was also not spotted in *Birthday Party Decorations* and *Bricks*, yet fewer than 10% of the Express students claimed to find it the most helpful. In *High Chairs* and *Christmas Party Decorations*, the three students who used the *figure-ground reversal* strategy in the test also believed this strategy would greatly help them in rule construction.

Students who had produced a correct functional rule through *guess-and-check* or an *indeterminate* strategy appeared to find the *constructive* and *reconstructive* strategies most helpful, with a strong preference for the *reconstructive* strategy in *Birthday Party Decorations* (41%) and *Bricks* (42%), and the *constructive* strategy in *High Chairs* (34%) and *Christmas Party Decoration* (49%). The first strategy, together with *repeated substitution*, was also highly preferred by those unsuccessful students whose strategies were classified as *miscellaneous*. Close to 90% of these unsuccessful students opted for the *repeated substitution* and *constructive* strategies in *Birthday Party Decorations* and the values in *Bricks* and *Christmas Party Decorations* were about 80% and 60% respectively.

Table 5.12: Frequency of Express students' generalising strategies in JuStraGen and choices of best-help strategies in QBBS 1

		Birthday Party Decorations					Bricks				
	QBBS Strategy / JuStraGen strategy	RS	C	R	D	Total	RS	C	R	D	Total
Numerical	Repeated substitution	12	6	4	1	23	11	7	10		28
	Finding difference leading to recursive rule	24	22	9	7	62	14	16	15	3	48
	Others	5	1	6		12	5		6		11
	Total for numerical	41	29	19	8	97	30	23	31	3	87
	%	42	30	20	8	100	35	26	36	3	100
Figural	Constructive	18	46	63	10	137	9	75	30	1	115
	Reconstructive – Constructive	1	2	2		5		1	3		4
	Reconstructive – Figure-ground reversal			1		1			2		2
	Deconstructive										
	Figure-ground reversal		1	2		3		1			1
	Total for figural	19	49	68	10	146	9	77	35	1	122
	%	13	34	46	7	100	7	63	29	1	100

Guess-and-check (GC)	Guess-and-check	3	3	3		9	3	7	3	3	16
Indeterminate (I)	Indeterminate	13	14	22	3	52	14	18	30	1	63
	Total for GC and I	16	17	25	3	61	17	25	33	4	79
	%	26	28	41	5	100	21	32	42	5	100
Miscellaneous	Miscellaneous	19	10	3	1	33	19	18	12		49
	%	58	30	9	3	100	39	37	24		100
	Grand total	95	105	115	22	337	75	143	111	8	337

RS: repeated substitution, C: constructive, R: reconstructive, D:deconstructive

Table 5.13: Frequency of Express students' generalising strategies in JuStraGen and choices of best-help strategies in QBBS 2

		High Chairs					Christmas Party Decorations				
	QBBS Strategy / JuStraGen strategy	RS	C	R	FGR	Total	RS	C	R	FGR	Total
Numerical	Repeated substitution	9	5	3	2	19					
	Finding difference leading to recursive rule	25	12	13	7	57	11	6	4	6	27
	Others	8		4	1	13	1				1
	Total for numerical	42	17	20	10	89	12	6	4	6	28
	%	47	19	23	11	100	42	21	16	21	100
Figural	Constructive	7	68	32	7	114	6	74	52	15	147
	Reconstructive – Constructive	2	5	17	3	27			10	1	11
	Reconstructive – Figure-ground reversal			12	1	13			4		4
	Figure-ground reversal				2	2				1	1
	Constructive – Figure-ground reversal		1	2		3					
	Total for figural	9	74	63	13	159	6	74	66	17	163
	%	6	46	40	8	100	4	45	40	11	100

Guess-and-check (GC)	Guess-and-check	3	1	6	2	12	4	3	3		10
Indeterminate (I)	Indeterminate	7	16	8	7	38	4	19	7	5	35
	Total for GC and I	10	17	14	9	50	8	22	10	5	45
	%	20	34	28	18	100	18	49	22	11	100
Miscellaneous	Miscellaneous	8	10	14	7	39	30	33	16	22	101
	%	20	26	36	18	100	30	33	16	21	100
	Grand Total	69	118	111	39	337	56	135	96	50	337

RS: repeated substitution, C: constructive, R: reconstructive, FGR:figure-ground reversal

For the Normal (Academic) students, a cross tabulation of the frequencies of their generalising strategies in the *JuStraGen* test and their *QBBS* choices for *Birthday Party Decorations* and *Bricks* was provided in Table 5.14, and the frequencies for *High Chairs* and *Christmas Party Decorations* in Table 5.15. The students achieved several similar outcomes to those for the Express students. For instance, the *constructive* and *reconstructive* strategies were well regarded as the most helpful by a vast majority of figural strategy users in the test, both attaining a combined percentage of over 75% in each *QBBS* task. Another similarity was the popularity of the *constructive* strategy amongst those who had used it in the test as well as the numerical strategy users. Taking the *Christmas Party Decorations* survey task as an example, 11 out of the 16 users of the *constructive* strategy opted for the same strategy in the survey. On the other hand, the *deconstructive* and *figure-ground reversal* strategies were the least popular in comparison to the other strategies amongst the figural strategy users. Interestingly, only one Normal (Academic) student had used the *figure-ground reversal* strategy, in combination with the *reconstructive* strategy, to formulate the functional rule for *High Chairs* in the test, yet the *figure-ground reversal* strategy was deemed as the most helpful by a few of them: four out of 22 in *High Chairs* and one out of 18 in *Christmas Party Decorations*.

Besides the similar outcomes highlighted above, Normal (Academic) students who had produced a correct functional rule through *guess-and-check* or an *indeterminate* strategy also appeared to find *constructive* and *reconstructive* strategies most helpful, in much the same way as their Express counterparts. The *reconstructive* strategy was particularly popular in *Birthday Party Decorations* (38%) and *High Chairs* (57%) and the *constructive* strategy in *Bricks* (40%) and *Christmas Party Decorations* (50%). The outcomes for unsuccessful students whose strategies fell under the *miscellaneous* category paralleled those for the corresponding Express students only in two tasks: *Birthday Party Decorations* and *Bricks*, which comprised over 75% of the unsuccessful students in the *repeated substitution* and *constructive* strategies combined. However, for *High Chairs* and *Christmas Party Decorations*, the four approaches offered in the task were equally preferred by the unsuccessful students, an outcome different from that for the unsuccessful Express students.

Finally, Normal (Academic) students who had produced a correct recursive rule favoured the *constructive* strategy over *repeated substitution* in *Birthday Party Decorations*, *Bricks* and *Christmas Party Decorations*. There were an equal number of students selecting each of the two strategies in *High Chairs*. This result stood in contrast to the preference for *repeated substitution* amongst those Express students who had generated a recursive rule correctly.

Table 5.14: Frequency of Normal (Academic) students' generalising strategies in JuStraGen and choices of best-help strategies in QBBS 1

Normal (Academic)	QBBS Strategy / JuStraGen strategy	Birthday Party Decorations RS	C	R	D	Total	Bricks RS	C	R	D	Total
Numerical	Repeated substitution		1			1	1	1			2
	Finding difference leading to recursive rule	33	44	8	10	95	29	48	14	1	92
	Others			1		1				1	1
	Total for numerical	33	45	9	10	97	30	49	14	2	95
	%	34	46	9	11	100	32	51	15	2	100
Figural	Constructive	2	5	5		12	2	7	6		15
	Reconstructive – Constructive										
	Reconstructive – Figure-ground reversal										
	Deconstructive				1	1					
	Figure-ground reversal										
	Total for figural	2	5	5	1	13	2	7	6		15
	%	16	38	38	8	100	13	47	40		100

Guess-and-check (GC)	Guess-and-check	1		1		2	1		1		2
Indeterminate (I)	Indeterminate	3	4	5	2	14	2	6	2	3	13
	Total for GC and I	4	4	6	2	16	3	6	3	3	15
	%	25	25	38	12	100	20	40	20	20	100
Miscellaneous	Miscellaneous	17	29	4	2	52	15	24	7	6	52
	%	32	56	8	4	100	29	46	13	12	100
	Grand total	56	83	24	15	178	50	86	30	11	177

RS: repeated substitution, C: constructive, R: reconstructive, D:deconstructive

Table 5.15: Frequency of Normal (Academic) students' generalising strategies in JuStraGen and choices of best-help strategies in QBBS 2

Normal (Academic)	QBBS Strategy / JuStraGen strategy	High Chairs RS	High Chairs C	High Chairs R	High Chairs FGR	High Chairs Total	Christmas Party Decorations RS	Christmas Party Decorations C	Christmas Party Decorations R	Christmas Party Decorations FGR	Christmas Party Decorations Total
Numerical	Repeated substitution				1	1					
	Finding difference leading to recursive rule	34	33	23	14	104	7	14	3	7	31
	Others										
	Total for numerical	34	33	23	15	105	7	14	3	7	31
	%	32	31	22	15	100	23	45	10	22	100
Figural	Constructive		9	5	3	17		11	4	1	16
	Reconstructive – Constructive		1	2	1	4			2		2
	Reconstructive – Figure-ground reversal			1		1					
	Figure-ground reversal										
	Constructive – Figure-ground reversal										
	Total for figural		10	8	4	22		11	6	1	18
	%		46	36	18	100		61	33	6	100

Guess-and-check (GC)	Guess-and-check		1	2	1	4		2	1		3
Indeterminate (I)	Indeterminate			2	1	3	1	1	1		3
	Total for GC and I		1	4	2	7	1	3	2		6
	%		14	57	29	100	17	50	33		100
Miscellaneous	Miscellaneous	12	10	12	9	43	31	36	26	30	123
	%	28	23	28	21	100	26	29	21	24	100
	Grand Total	46	54	47	30	177	39	64	37	38	178

RS: repeated substitution, C: constructive, R: reconstructive, FGR:figure-ground reversal

5.2.2.1 Summary and discussion

Amongst students who engaged a numerical strategy in the test, the *repeated substitution* strategy was the most popular of the four choices for the Express students. Although many of them adhered to the numerical approach, the majority, on the basis of considering the frequencies of the three figural approaches collectively, did not seem to find their numerical strategies any helpful. As a result, they abandoned their strategies to opt for a figural strategy, typically the *constructive* or *reconstructive* approach. The *deconstructive* and *figure-ground reversal* approaches were the least popular. Interestingly, for the numerical strategy users in the Normal (Academic) course, their clear favourite was the *constructive* strategy, then followed by the *repeated substitution* strategy, which was completely the reverse of the Express students'. This finding might offer some insight for the low frequency of *repeated substitution* for the Normal (Academic) students in the written test. Given that the strategy is advocated in the local mathematics textbooks and is also believed to be a popular approach taught by the local mathematics teachers (Chua & Hoyles, 2010b), there is sufficient reason to assume that the Singapore students have been exposed to it. The fact that many Normal (Academic) students neither employed it nor judged it as the most helpful strategy suggests that the students might still not have appreciated its process, so it was not used.

The top two choices of best-help strategy amongst the figural strategy users in both student courses were the *constructive* and *reconstructive* strategies, thus showing fairly well the consistency between the type of generalising strategy they used in the test and the type of strategy they judged as helpful in the survey. Although the use of the *reconstructive* strategy was rather infrequent in the test, it was remarkable to learn that such a strategy had gained popularity amongst the students. The other two figural approaches involving the *deconstructive* and *figure-ground reversal* strategies were relatively unpopular. Not surprisingly, some students, despite employing a figural strategy in the test, decided to ditch it in favour of a numerical strategy, but the number of such students was somewhat small.

The comparison of the generalising strategies engaged by users of *guess-and-check*, *indeterminate* and *miscellaneous* strategies with their choices of best-help strategies did not seem to reveal any clear pattern of their preferences. Their top favourite inclined towards the *constructive*, *reconstructive* and *repeated substitution* strategies. Again, the *deconstructive* and *figure-ground reversal* strategies were unpopular.

5.2.3 EFFICACY OF STUDENTS' CHOICES OF BEST-HELP STRATEGIES

This section seeks to examine whether students were able to formulate the functional rule of a pattern using the best-help strategies that they had judged as the most helpful in the survey. The sub-research question is stated below:

> 4.5 What is the efficacy of the students' choice of best-help generalising strategies on their rule construction?

Table 5.16 presents the outcomes of students' rule construction using the generalising strategies that they judged as most helpful. Overall, five Express and three Normal (Academic) students were able to establish the correct functional rule without any assistance whereas another two Express and three Normal (Academic) students needed some guidance to produce the rule correctly. Only three students, one Express and two Normal (Academic), were totally unsuccessful despite the promptings from the researcher.

Table 5.16: Interviews outcomes of rule construction using best-help generalising strategies

Generalising strategies in *QBBS* tasks	Express			Normal (Academic)		
	BD		Xmas	BD		Xmas
Repeated substitution	64M3, P		77M3, U	217M1, U		136M2, U
Constructive	15M1, S		44M2, S	148M2, P		138M3, S
Reconstructive	37M1, S		75M2, P	160M2, S		141M2, P
Deconstructive	11M2, S			153M3, S		
Figure-ground Reversal			93M1, S			149M1, P
Rule construction outcome	S	P	U	S	P	U
	5	2	1	3	3	2

BD: Birthday Party Decorations; Xmas: Christmas Party Decorations
S: successful without prompting; P: successful with prompting; U: unsuccessful

For *Birthday Party Decorations,* only one of the four Express students required guidance to formulate the linear rule whilst the remaining three did not need any. However, for *Christmas Party Decorations*, only two students managed to yield the correct quadratic rule on their own and of the remaining two students, one was guided to produce the rule by means of the *reconstructive* strategy and the other failed to generate the rule using the *repeated substitution* strategy.

The achievements of the Normal (Academic) students followed a similar trend to those of the Express students, with fewer students working out the correct rule on their own in the quadratic task than in the linear task. Of the three students who developed the *Birthday Party Decorations* rule, two did not require any guidance. The student who picked *repeated substitution* as the most helpful strategy did not accomplish the construction of the rule. In contrast, only one student was able to derive the *Christmas Party Decorations* rule using the *constructive* strategy without any help. Given that the quadratic rule was not straightforward to develop using the *repeated substitution* strategy, it was hardly surprising that the student who had judged it as most helpful failed to work out the rule.

5.2.3.1 Interview vignettes

The interviews of eight students from each course offered a first glimpse into a few aspects of students' generalisation of pattern, including, for instance, how successful students developed their functional rules, how students who picked the *repeated substitution* strategy in the quadratic task performed in rule construction, what impeded unsuccessful students' construction of a functional rule, and why students liked or disliked the various generalising strategies. What follows are discussions of these aspects supported by evidence drawn from the student interviews.

(1) How successful students developed their rules

Figure 5.1 provides the pictorial representation of the *figure-ground reversal* strategy for *Christmas Party Decorations*. Student 93M1 chose it as the most helpful strategy and produced the correct rule, $(n+2)^2 - [2 \times (n+1)]$. To generate this rule, she knew "*the size number has something to do with the rule*", a point she mentioned during the interview.

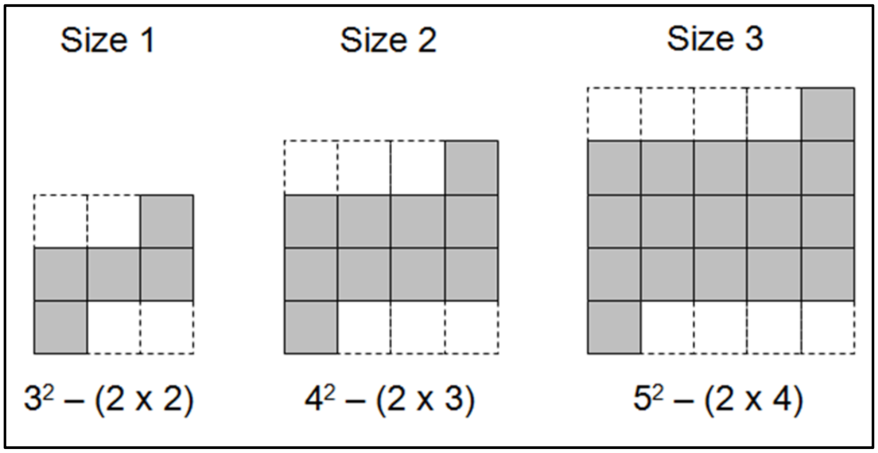


Figure 5.1. Pictorial representation of figure-ground reversal in *Christmas Party Decorations*

The following transcript of the interview, conducted by the researcher (R), with Student 93M1 illustrates how she established the rule using the position–to–term relationship.

> 93M1: I see the 3^2 first. Then it's size number plus two. Then this one also (pointing to the Size 2 configuration).
> R: Size 2 also what?
> 93M1: It's 2 plus…the 4^2 is also size 2 plus another 2.
> R: You mean the number 4 is actually…
> 93M1: This one (pointing to 4 in the numerical statement, $4^2 - (2 \times 3)$, given beneath the Size 2 configuration). Number 4 is size 2 plus 2. Squared.
> So then this one (pointing to the 2 in all three given numerical statements) is all the same, so there is no need to change.
> This one (pointing to the rightmost numbers in the numerical statements) is size number plus 1. All are the same. So it's like that.

Two manifestations of an understanding of this position–to–term relationship were observed in the interviews with Student 11M2 and Student 64M3. Student 11M2 favoured the *deconstructive* strategy over the others in *Birthday Party Decorations* and developed a functional rule successfully, by linking the size number to the terms. Figure 5.2 presents the pictorial representation of this strategy.

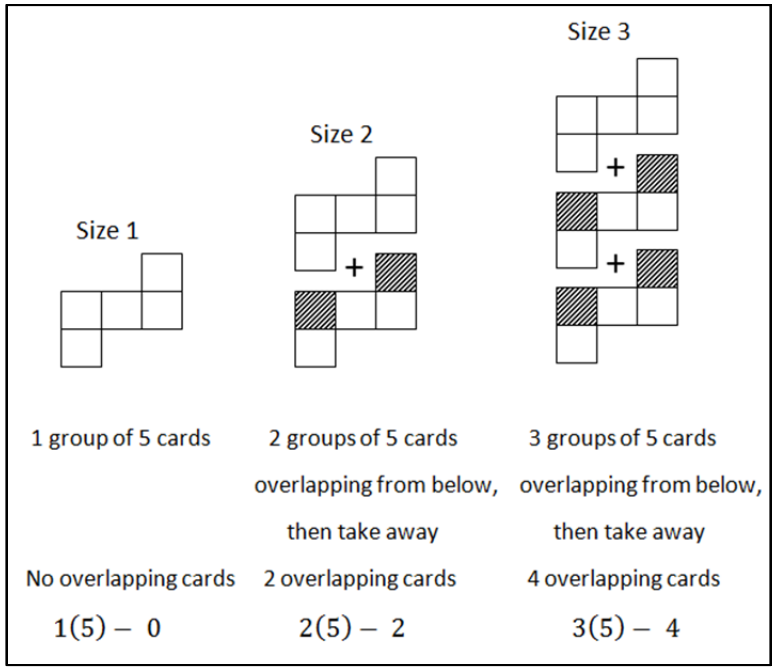


Figure 5.2. Pictorial representation of deconstructive in *Birthday Party Decorations*

The transcript below describes how Student 11M2 worked out the rule.

R: I want you to imagine you were this student. If you work out the pattern this way, what would the rule be? Can you write it out for me?

11M2: (she paused a little, then wrote down $n(5) - (n - 1) \times 2$ on the questionnaire) Is it like that? I think it's like that.

R: How do you know this is right or wrong?

11M2: I think it's like over here (pointing to 1 in 1(5) – 0 and the first 2 in 2(5) – 2, then circling the size numbers in Size 1 and Size 2) the number here are the same as the number of the size.
Then over here (now pointing to 0 and the second 2 in the numerical statements) the minus is like if you minus 2 by 1 (using Size 2 as a specific example) then times 2 is 2.

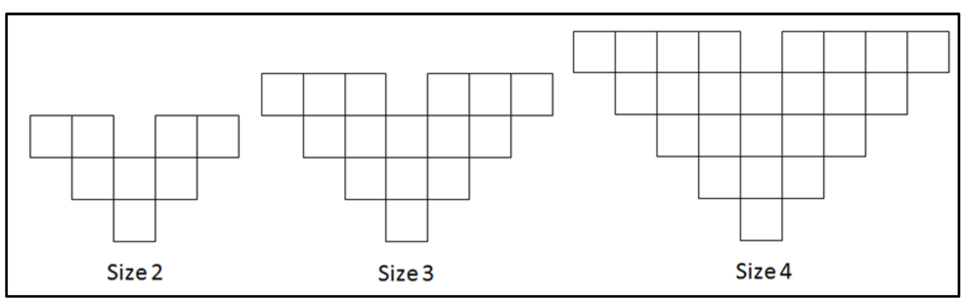


Figure 5.3. Tulips configurations

Figure 5.3 presents the successive *Tulips* configurations as shown in the *JuStraGen* test. Student 64M3 was asked to elaborate her incomplete rule for this task first, then to make an attempt to derive a functional rule. She first counted the number of tiles in each of the three configurations and then wrote down the values 8, 15 and 24 for Sizes 2, 3, and 4 respectively on the test script. After a long pause, the following interaction in the interview transcript took place.

> 64M3: The size times... *n* times *n* plus 2 (as she wrote $n \times (n + 2)$).
> R: How did you figure out this rule?
> 64M3: One by one. Like 2 times 4 (pointing to 8 on the test script) for Size 2, then 3 times 5, 4 times 6.
> R: Ok. Why did you take 2 times 4, 3 times 5, 4 times 6?
> 64M3: This 24 is the total number (circling Size 4 with a pen).
> R: For 24, why didn't you take 3 times 8 or 2 times 12?
> 64M3: Because size is *n*. Then take this as the main one.

By "taking this as the main one", Student 64M3 meant to use *n* to generate the general term.

(2) How students who judged repeated substitution as most helpful responded in interviews

Of the four interviewees who judged *repeated substitution* as the best-help strategy, three were unsuccessful and one needed guidance to derive the linear rule. Student 217M1, one of the unsuccessful students, mentioned during the interview that she was weak in Mathematics and poor in spatial visualisation. She picked the *repeated substitution* strategy in *Birthday Party Decorations*, claiming she understood how the subsequent terms were obtained and, yet, she was unable to establish the rule for the general term. The transcript below shows her response.

> R: I want you to imagine, if you were the student, how would you get the rule from here?
> 217M1: (pause) Can use the formula, right?
> R: How? Can you show me?
> 217M1: (long pause) Still don't know how to do it but I understand the method, but don't know how to get the rule.
> R: So is this method helpful to you?
> 217M1: Because I get the answer.
> R: What answer? How would that help you to get the answer?
> 217M1: (long pause) I really cannot get the rule thing.

As Student 217M1 appeared to be very stressed, the researcher decided not to pursue this task any further, thus explaining why she was not offered any guidance to develop the rule.

Since the construction of a quadratic rule using the *repeated substitution* strategy was not simple, it was not surprising that Student 136M2 and Student 77M3 had extremely limited success in working out the functional rule for *Christmas Party Decorations* (see the shaded portions in Figure 5.1 for the original configurations). The following transcript shows that Student 136M2 conjectured a rule which she later realised was wrong.

> R: Can you find the rule for me?
> 136M2: (long pause as she tried to figure out a rule)
> n equals to n plus n plus 2 (writing $n = n + n + 2$ as she articulated)
> R: If it is Size 1, it will be 1 plus 1 plus 2. Correct?
> So you are saying there are 4 cards in Size 1. But we know there are 5 cards.
> What do you think of your rule?
> 136M2: (long pause) That one is wrong.

The given rule, although it was incorrect, indicates that Student 136M2 recognised the constant increase of 2 in the first difference.

The transcript of Student 77M3 sheds some light on why some students had judged the method applying the *repeated substitution* strategy as most helpful. According to the student, this method allowed him to spot the change in the pattern directly, a reason somewhat similar to Student 217M1's.

> R: Based on this, what would the rule be?
> 77M3: (pause) I don't know. Don't know.
> R: So why did you choose this method?
> 77M3: Because it's quite clear and easy to see.
> R: Ok. Clear to you, easy to see, but then it is not straightforward.
> 77M3: But then I see the behind because it's the everything, then plus 5, plus 7, plus 9.
> (referring to the first differences between pairs of consecutive terms)

(3) Potential factors hindering students' success in making generalisations

A possible reason for the moderate to high failure rates in quadratic generalising tasks might be due to students having limited exposure to these tasks during their mathematics lessons. Although one might have hoped that students would be able to build on what they have learned previously with linear generalising tasks to a new task, transfer of knowledge does not always happen. As a result, students might not know how to handle such tasks when they encountered them. The transcript of 217M1 highlights this problem.

> R: Why is this question [Oh Deer!] difficult?
> 217M1: Because usually a question…from Size 2 to Size 3, you need to add a standard number.
> From Size 2, you add 8. Then you will get Size 3. Then from Size 3 to Size 4, you

should also have plus 8 also.
But in this situation, it's like plus 8, then plus 10.
So for me, it's like I was stuck there.

R: I see. So you don't see a common difference…

217M1: like there is no pattern.

R: Ah…there is no pattern. So that's how you find this question difficult.

Unsuccessful students appeared to share a common problem. They used the letter *n* in their rules, yet they were unclear of its meaning. The following two transcripts of Student 149M1 and Student 64M3 illustrate this point.

R: Do you know what *n* represents in the rule?
(Student shook his head.)
Have you learnt this before?

149M1: Yes.

R: So *n* actually represents…

149M1: Constant number

R: (referring to the student's rule) This will actually tell you the number of cards in Size n.
So what do you think n represents?

149M1: A number.

R: What number?

149M1: the same number throughout.

R: [What's the] general rule? (inviting Student 64M3 to develop the rule using her best-help strategy)

64M3: 5 plus 3 (long pause)

R: General rule is for Size *n*. If I want to find Size *n*…

64M3: *n* plus 3.

R: How do you know if this is right?

64M3: (pause) Don't know.
(pause) 5 plus 3 is 8. Then 8 plus 3 is 11.

R: What does n represent?

64M3: Represent the number before adding.

Student 149M1 only knew the letter *n* represented a number whereas Student 64M3 took it to mean the previous term. Clearly, the students were ignorant of the meaning of *n* as used in the symbolic rule.

The case of Student 136M2, for *Christmas Party Decorations* involving only Sizes 1 and 4, showed that when numerical cues are disconnected from the figural cues, a wrong pattern might be produced. Consider the interview transcript below.

R: In this question, Size 2 and Size 3 are missing, but you actually said there are 12 cards in Size 2 and 19 cards in Size 3. Can you tell me how you came up with these values?

136M2: I find the size for the remaining two sizes, then after that I divide it by 3 to find what is the number to add on to get the size.

R: So you actually find the difference between 26 and 5 (these are the number of cards in Size 1 and Size 4), then you divide it by 3.

Student 136M2 conjectured that the difference between each consecutive pair of configurations was constant and, upon recognising the difference between the two given configurations was 26, divided this value evenly over four successive configurations. This conjecture eventually led to working out the terms 12 and 19 for Sizes 2 and 3 respectively. Although the numerical terms [5, 12, 19, 26] formed a linear sequence, they did not match the visual cues of the figural pattern and the validity of the numerical pattern was verified by getting the student to draw out the configurations for Sizes 2 and 3. Later in the interview, she realised that her configurations for Sizes 2 and 3 "cannot form the same shape" as the two given configurations.

(4) Reasons for choosing or rejecting a particular generalising strategy

Students picked their favourite generalising strategies based on many reasons, including, for instance, their prior experience and personal opinion. Consider Student 141M2 and Student 138M3. Student 141M2 chose the *reconstructive* strategy because he had learnt it previously in primary school, and Student 138M3 found the idea of breaking up the whole configuration into parts using the *constructive* strategy easy for most students to understand. However, some students such as Student 153M3 and Student 93M1 found breaking up the configurations then piecing them back later in order to develop a functional rule rather confusing. Student 77M3 did not choose this strategy to avoid having to work with the squaring of terms.

Similarly, whilst some students such as Student 64M3 found the *repeated substitution* strategy easier and less complicated than the other approaches, those, for instance, Student 93M1, who eschewed such a method generally found it time consuming, confusing and tedious to set up a table of values. Student 138M3 felt that this numerical approach did not permit him to notice the explicit link between the size number and the number of cards used in *Christmas Party Decorations*.

The *deconstructive* strategy was judged as the least helpful because students found the overlapping and subtraction complicated and confusing to understand. In addition, some students such as Student 37M1 and Student 64M3 were also concerned about overlooking the subtraction of the overlapping cards and miscounting them. In a similar vein, the *figure-ground reversal* was also not popular amongst the students because some worried about subtracting more tiles or cards than required.

5.2.3.2 Summary and discussion

Half of the 16 students derived the rule unaided and another five needed some guidance to develop the rule during the interviews. Many of these students, especially those in the Normal (Academic) course, did not formulate a correct functional rule previously in the *JuStraGen* test. Hence, the numerous successes of a significant number of students in rule construction using their choice of best-help strategies suggest that their preferred strategies were indeed helpful. This gives confidence to the validity of the survey results.

The two students who chose *repeated substitution* as most help for the quadratic tasks failed to develop a correct functional rule as anticipated because the process was rather advanced for Secondary Two students. The unsuccessful attempt of the Normal (Academic) student who picked this strategy was a little surprising given that it was supposed to be a familiar approach for them. Another surprise sprang from the successes of the two students who opted for the *deconstructive* strategy, which was thought to be an abstruse approach for many to deal with.

Most successful students appreciated the meaning of the letter *n* to be used in the rule and would examine the size numbers and the corresponding terms in search of a relationship between them. Some did not understand its significance even though they produced a correct symbolic rule. In contrast, students who failed to establish a correct rule appeared to be ignorant of what the letter *n* represented. As long as this meaning is not recognised, the position–to–term relationship is usually obscured.

The case of Student 136M2 mistaking the *Christmas Party Decorations* rule for a linear relationship is not totally unexpected especially when it is a common practice amongst students to convert a figural pattern to a numerical sequence, then followed by using the resulting numerical sequence to make generalisations. So when students do not further rely on any visual cues to ascertain the validity of the numerical sequence, a disconnect between the two forms of pattern exists, and errors such as Student 136M2's can happen. Therefore, her problem is not caused by any design flaw in the task but by the student herself for making a wrong conjecture about the pattern and using an inappropriate strategy (i.e., finding the common difference). In fact, this case is a prime example highlighting a crucial aspect of transforming a figural pattern into a numerical sequence: that is, it is imperative to check the visual cues carefully to confirm the correctness of the numerical sequence.

On the same issue of students misinterpreting the underlying pattern structure, a few similar cases were also spotted in the *JuStraGen* test. Another prime example is Student 22M2's interpretation of the single-configuration *Bricks* pattern that he was assigned to work on: the number of rows in a configuration corresponds to its size number, and the number of bricks per row alternates between four in odd rows and three in even rows. This student's error is really unfortunate despite providing a brief description of the configuration in the task to make clear that each *Bricks* configuration consists of only three rows. Therefore, it is maintained again that the error is not triggered by any design flaw in the *Bricks* task. Rather, the problem lies with the student not reading the provided description of the configuration carefully.

The interviews throw light on a number of reasons influencing how students decide on the best-help strategy. Apart from personal opinion and prior learning experience, other reasons range from a dislike for working with mathematical operations such as subtraction and squaring to finding certain procedures inherent in the approaches, such as overlapping and breaking up shapes, as well as setting up a table of values, confusing.

5.3 SUMMARY OF STUDY II

This chapter has reported the noteworthy findings of Study II. There are four main conclusions. First, the *constructive* strategy was the clear favourite of the Express and Normal (Academic) students. Contrary to the researcher's initial belief, *repeated substitution* was rather popular amongst the Normal (Academic) students but not the Express students. Second, there were no significant differences in the distribution of student choice of best-help strategies between (a) Express and Normal (Academic) students working with successive pattern format, and (b) working with successive and non-successive pattern formats in both courses. Third, most Express students who employed a numerical strategy in the *JuStraGen* test continued to opt for a numerical approach in the survey whereas numerical strategy users in the Normal (Academic) course tended to choose a figural approach. Those who used a figural strategy in the test favoured either the *constructive* or *reconstructive* strategy in the survey. Lastly, the majority of students selected for the interviews experienced success when asked to construct a functional rule using their choice of best-help strategies. Two valuable insights gleaned from the interview data were the benefit of appreciating the significance of the variable used, and the importance of utilising visual cues when developing the functional rule for a figural pattern.

CHAPTER 6: CONCLUSIONS, LIMITATIONS AND IMPLICATIONS

This chapter begins with a summary of the present study, followed by a discussion of its significant contributions in relation to prior research, alongside a comparison of Singapore students with those in other countries. Then it presents the limitations of this study and suggests teaching strategies for teachers and teacher educators to help them improve their teaching of pattern generalisation. Finally, the chapter proposes some suggestions for future research studies in pattern generalisation.

6.1 SUMMARY OF THE STUDY

The present study investigated Singapore secondary school students' generalisations and justifications of patterns, beliefs of a best-help generalising strategy and the influence of two task features, namely, the format of pattern display and the type of functions, on their rule construction. The following four main research questions were proposed.

1. How do Singapore secondary school students establish the rule that defines a figural pattern?
2. How do Singapore secondary school students justify the rule they constructed?
3. How do task features influence Singapore secondary school students' rule construction?
4. What do Singapore secondary school students judge to be the most helpful generalising strategy for constructing the functional rule?

To answer these questions, two studies, Study I and Study II, were implemented. Study I, an investigation of students' generalisations and justifications, as well as the impact of two task features on the two aspects above, addressed the first three main research questions. It comprised a comprehensive overview of the current state of Singapore Secondary Two Express and Normal (Academic) students' generalisations and justifications and an investigation of the effects of the format of pattern display and the type of functions. Study II, an exploration of what the students judged to be a best-help generalising strategy, addressed the fourth main research question and considered their choice of best-help strategies and its relationship with their generalising strategies.

The data for Study I was collected through administering a specially designed and validated paper-and-pencil test – the *JuStraGen* test, consisting of eight generalising tasks to 515 Secondary Two Express and Normal (Academic) students from three secondary schools. The students were divided into two groups according to their scores on a baseline test, their mathematics grades in a national examination in Singapore, and their gender, with each group assigned to a different pattern format. Details of the distribution of students by schools, *GAT* scores, *PSLE* mathematics grades and gender show that the students chosen were as representative as possible in terms of mathematical ability.

The data for Study II was collected through administering to the same group of 515 students a specially devised and validated paper-and-pencil questionnaire – the *QBBS* instrument, comprising four tasks. The students were asked to identify their choice of best-help generalising strategy. From the entire student sample, 16 students, consisting of eight from each student course, were chosen for interviews to assess the efficacy of their choice of best-help generalising strategy selected in the *QBBS* survey.

The key findings of each study are now summarised below.

6.1.1 KEY FINDINGS OF STUDY I

The key findings of Study I are listed below:

(1) At least half of the Express students constructed a correct functional rule. This contrasted with the results from the Normal (Academic) students where no more than 15% established a correct functional rule in each task.

(2) Success rates were higher in linear tasks than in quadratic tasks for both Express and Normal (Academic) students.

(3) Express students developed a diverse range of equivalent functional rules for both linear and quadratic generalising tasks compared with Normal (Academic) students who produced a rather limited variety of equivalent functional rules because of the very low success rates.

(4) The expression, $n + k$, as the symbolic rule for linear generalising tasks, was not common amongst the Express and Normal (Academic) students.

(5) Three modes of representing a functional rule were identified: *in notations*, *in words* and *in alphanumeric form*. The first type was the most prevalent mode for all successful students and this was then followed by expressing the rule in alphanumeric form.

(6) Four categories of generalising strategies were established: *numerical*, *figural*, *guess-and-check*, and *indeterminate*. The *figural* strategies were the most predominant amongst the Express students whereas the frequencies of *figural* and *indeterminate* strategies were high amongst the Normal (Academic) students.

(7) Five types of numerical strategies were observed: *repeated substitution*, *comparison*, *substituting values into formula*, *solving equations*, and *grouping* – a new strategy in the literature. The use of the first two strategies was widespread amongst the Express students; however, the most widely used numerical strategy by the Normal (Academic) students could not be inferred in this study.

(8) Nine types of figural strategies were found, the *constructive* strategy being the most popular strategy amongst all the students.

(9) Some students used a combination of figural strategies to construct the functional rules. Six types of *combo* strategies were identified: *constructive–comparison*, *constructive–reconstructive*, *constructive– figure-ground reversal*, *reconstructive–constructive*, *reconstructive– figure-ground reversal*, and *figure-ground reversal–reconstructive*. Apart from *reconstructive– figure-ground reversal*, the remaining five combinations are new in the literature.

(10) The frequencies of *guess-and-check* and *indeterminate* categories were fairly high in both courses.

(11) Four categories of justification schemes were identified: *justifying functional rules without diagram*, *justifying functional rules with diagrams*, *justifying*

recursive rules, and *miscellaneous*. The first two categories were the most prevalent for Express students. Similar findings were observed for Normal (Academic) students, but the frequency of *miscellaneous* schemes was high in certain tasks as well.

(12) Eight types of justification schemes were found under the category of justifying functional rules without diagram. One of the schemes was the empirical verification of the validity of a rule which had high frequency amongst all the students.

(13) Three types of justification schemes were found under the category of justifying functional rules with diagrams: *providing a generic configuration*, *providing a few configurations*, and *providing a few configurations and organising numerical values in a tabular form*. The first two types were very commonly used by all the students.

(14) Generalising tasks involving non-successive configurations were significantly more demanding than those with successive configurations for the Normal (Academic) students, but not the Express students.

6.1.2 KEY FINDINGS OF STUDY II

The key findings of Study II are listed below:

(1) The clear favourite of the Normal (Academic) students was the *constructive* approach whilst the Express students highly favoured the *constructive* and *reconstructive* approaches. The *repeated substitution* approach was popular amongst the less academic students, but not the more academic ones.

(2) The top choice of best-help strategy amongst the Express students who used a numerical strategy in the *JuStraGen* test was the *repeated substitution* approach, whereas it was the *constructive* approach for those in the Normal (Academic) course.

(3) Amongst the Express and the Normal (Academic) students who employed a figural strategy, their top favourites were the *constructive* and *reconstructive* strategies.

(4) 13 out of the 16 students selected for the interviews established a correct functional rule using their choice of best-help strategy. Of these 13, eight of them produced the rule on their own without any assistance whilst the remaining five also succeeded after some guidance.

6.1.3 CONCLUSIONS OF THE STUDY

The more academic students were successful with linear generalising tasks but floundered when they worked with quadratic tasks whereas the less academic students failed completely in both types of tasks. Successful students in both courses perceived the patterns in numerous ways and constructed a wide range of functional rules, expressed prevalently in notations, to describe them. They employed a variety of generalising strategies, some of which were novel in the literature, but their clear favourite were the figural strategies, as evidenced in both the test and the survey. Task features such as the format of pattern display and the type of functions could contribute to student difficulties and hinder their generalisations. Therefore they should be taken into account when examining the factors of student difficulties in making generalisations. Finally, most successful students justified their rules figurally and non-figurally, although some failed to justify correctly using an appropriate approach. Thus a successful generalisation is not always accompanied by an adequate correct justification.

6.2 SIGNIFICANT CONTRIBUTIONS TO THE FIELD

Pattern generalisation is a well-researched field and the present study contributes to the existing body of work in this field in several ways. The significant contributions are presented below.

First, most studies on pattern generalisation have been undertaken in the west, offering a vast knowledge of students' generalising abilities and generalising strategies. However, there are few studies, reported in English, in the literature involving students in the East Asia, in particular those from the top-performing countries in mathematics. Consequently, little is known about their generalising abilities, thinking and generalising strategies. This study provides new insight into the types of rules Singapore secondary school students produce, the types of generalising strategies and justification schemes they employ, and their beliefs of what they would judge as the most helpful generalising strategy for rule construction. Such rich data are valuable as they not only deepen one's understanding of how East Asian secondary school students visualise, think and reason about the pattern structure but also facilitate comparisons with previous findings already reported in the literature. Some notable research achievements in the present study will now be highlighted.

One of the most striking findings from the *JuStraGen* test is the numerous ways of visualising the patterns and the wide variety of functional rules established. The more academic students are particularly proficient in pattern discernment and rule construction in both linear and quadratic generalising tasks. Although their achievements in the linear tasks are remarkable and consistent with previous results in the GCE "O" level examinations and TIMSS studies, it is their successes in the quadratic tasks that are more noteworthy given that (1) the patterns used in the test are novel and not the typical *square* or *triangle* numbers, and (2) such tasks are usually tough for students across different age groups, including, for instance, 7th graders in the US (see Steele, 2008), 8th and 9th graders in Lebanon (see Jurdak & El Mouhayar, 2014), and even adult learners in the US (see Rivera & Becker, 2007).

For the less academic students, the predominant interpretation of the patterns, whether linear or quadratic, is of the recursive relation. Only a small proportion of the students appreciated the requirement of the tasks and described the functional rules correctly. Plausible explanations for their poor performance in the test include ignorance of the type of rule to be produced, ignorance of the limitations of a recursive rule and the benefit of a functional rule, dislike for generalising tasks, and inexperience in dealing with generalising tasks. As a result, the success rates were poor even for linear generalising tasks, let alone the quadratic tasks. Of the academically less able student group, the successful students did not perceive the patterns in as many ways as did their more academic counterparts, so their variety of functional rules was narrower.

In contrast with studies in the west, the present study noticed that a vast proportion of Singapore students who found a correct functional rule in the written test had expressed it symbolically, with only a very small proportion describing the rule in words – see, for instance, Stacey and MacGregor (2001) who reported most Australian students wrote their rules predominantly in words. The fact that many students could describe the functional relationship for a pattern indicates that for them, the concept of variable was generally well understood. Data drawn from the student interviews support this argument. Again this is in contrast with other studies where it has been reported that the concept of variable is a stumbling block: for example, for US students in Becker and Rivera (2006). The Singapore students' prior experience with algebra might be a reason for their facility with symbolic functional rules. They had learnt to use letters to represent unknown values and the topic of number patterns before participating in the present study. Not surprisingly, they were able to establish more easily the general rule in the form of an algebraic expression.

Although successful students produced several different but equivalent functional rules that are, in Lee's (1996) language, *algebraically useful* (i.e., expressed in a form permitting the direct computation of any term), a few of these rules were disappointingly not meaningful. In other words, the functional rules could not be explained using the numerical or figural cues established from the pattern. They had been formulated through mere guessing, so the different components of the functional rule have little significance since they did not correspond to any part of a configuration.

Singapore students in ways similar to students from other countries shared common kinds of generalising strategies. Most successful students in Singapore often rely on figural cues obtained directly from the configurations presented in the patterns to work out their functional rules. Their top favourite figural strategy in the test involves breaking up the whole original configuration into non-overlapping parts and then deriving a functional rule by adding up the parts. The *QBBS* survey data confirm the popularity of this strategy and validate its frequent use in earlier studies, for instance, by Moss *et al*. (2008), Radford (2006), and Rivera and Becker (2008). Another approach employed by Singapore students as well as students in previous studies, for example, by Blanton and Kaput (2004), Rivera (2013), and Tanisli and Özdas (2009) is *repeated substitution*. Although this was the most popular numerical strategy in Singapore, this study indicated that it was judged favourably as a helpful strategy by the less academic students, but not by their more able counterparts. Chua and Hoyles (2010b) noted that secondary school mathematics teachers widely believe that *repeated substitution* is the most helpful strategy for the more academic students, but the current study demonstrates that the teachers' assumptions might be wrong.

When asked to pick the best-help generalising strategy in the *QBBS* survey, it is interesting to note that the more academic numerical strategy users chose *repeated substitution* out of the four given choices whereas the less academic numerical strategy users favoured the strategy involving breaking up a configuration into non-overlapping parts. For figural strategy users in both courses, they prefer to break up a configuration into non-overlapping parts or to rearrange the configuration to form something familiar.

Another marked observation to emerge from the comparison of generalising strategies noticed in the present and earlier studies is the revelation of a few new generalising strategies that are hardly described in the literature. These are the *grouping* and the *combo* strategies such as the *constructive–reconstructive* and *constructive–figural-ground-reversal* strategies. Therefore, this study adds new knowledge to the field by expanding the existing classification scheme of generalising strategies that students use to express generality in number patterns.

The present study has yielded a noteworthy observation which involves developing a functional rule for a figural pattern. It highlights the importance of relying on figural cues established from the configurations in rule construction. Using the numerical cues gathered from the number sequence converted from the figural pattern may lead to an erroneous conclusion about the pattern. The student interviews offer compelling evidence of students mistaking a quadratic pattern for a linear one when they did not connect the numerical and figural forms of the pattern.

Considerable insight into students' written justifications for pattern generalisation is provided as well. An analysis of the students' justification schemes shows that students do use approaches not included in Rivera and Becker's (2011) classification scheme. For instance, some students write down the numerical structures of the given configurations to demonstrate how the functional rule comes about. This way of justifying the rule does not make use of more examples other than the ones provided in the task, so it cannot be classified under *extension-generation*. Further, most successful students justify their functional rules with or without diagrams. For justification without using diagrams, empirical verification of the validity of a rule is a popular approach amongst the students, and this finding resonates with that of Lannin (2005), and Rivera and Becker (2011). It is important to note that such a justification does not explain how the rule is obtained. When diagrams are used, there are two common approaches: first, to draw just a particular configuration from which the generality is formulated; and second, to provide a few configurations to illustrate how the generality is developed. Lastly, some other successful students gave inappropriate justifications for their functional rules, which include transforming the modality of a functional rule from one form to another, and converting the figural pattern into a numerical sequence followed by indicating the changes between consecutive terms. The fact that students can construct a correct functional rule but cannot provide an appropriate justification indicates that the role and purpose of justifications is still not well understood.

The format of pattern display and the type of functions have long been suspected as potential stumbling blocks to students' poor performance in certain studies. One of the most intriguing findings of the present study, following a systematic and scientific probe, is the procurement of empirical evidence to confirm that these two task features do indeed contribute to student difficulties. Generalising tasks involving non-successive configurations are challenging for the less academic students, but their more able counterparts are not deterred by them. Despite being harder to work with, such tasks are found to promote different ways of visualising the pattern structure and foster functional thinking. It must be remembered that these results need to be interpreted with caution as they are indicative and not conclusive. Next, regardless of their academic achievements, students find quadratic generalising tasks more demanding than linear generalising tasks. However, it is interesting to note that their attempts to visualise the quadratic patterns in many ways and create a variety of functional rules have not been hampered by the nature of the tasks.

The literature review in Chapter 2 highlighted two generalising tasks that were poorly done. One of them was the border-tiling task, investigated by Hoyles and Küchemann (2001), which defeated a considerable number of high attaining Year 8 students in the UK, and the other was the TIMSS–2007 matchstick task that defeated a vast majority of students internationally (Foy & Olson, 2009). Some insight can now be offered for the poor student performance in these two tasks, which share a common feature: both depict the figural pattern using just a single configuration. Drawing on the current research finding about the effect of the format of pattern display, there is sufficient reason to believe that the use of a single configuration to display the figural pattern might have contributed to student difficulties.

Revisiting Steele's (2008) *Staircase* task discussed in Chapter 2, she had limited success in getting her US students to work out a functional rule for a classic quadratic task showing just a four-step-high staircase. Six of the eight students created the rule $n + (n-1) + (n-2) + (n-3) + \cdots + (n-n)$ using a recursive approach. Although Küchemann (2010) firmly maintains that the format of pattern display is a contributing factor to the student difficulties in this task, there is now strong reason to think that the student difficulties in constructing a functional rule in closed form are further confounded by the quadratic nature of the pattern. If the pattern display is now changed to a successive format, the "step" structure in the pattern might be highlighted even more when students who focus on the term-to-term relationship compare consecutive configurations. Thus the frequency of the abovementioned rule might be higher. The difficulty in quadratic generalising tasks is not experienced solely during rule construction; it can also occur when computing near and far terms. So the present study offers some insight to illuminate the unsuccessful attempts of the Lebanese students in a study by Jurdak and El Mouhayar (2014) to find both near and far terms in quadratic tasks.

Last but not least, the present study has produced and validated three new instruments, namely, *GAT*, *JuStraGen*, and *QBBS* to fill the missing gaps in pattern generalisation research since no appropriate instruments could be found in the literature that could assess

(a) a student's competence in pattern-related algebraic tasks, (b) the effect of the format of pattern display and the type of functions on their rule construction, and (c) students' beliefs of what they would judge as the most helpful generalising strategy for establishing a functional rule underpinning a pattern. In addition to being a new instrument, the *JuStraGen* test instrument comprises six new generalising tasks as well. These instruments not only provide a window for other studies on students' generalisations, they also serve as useful tools for other researchers in this field to take pattern generalisation research to greater heights.

6.3 LIMITATIONS OF THE STUDY

Any classroom-based study will inevitably have limitations beyond the researcher's control. This study is no exception and it may therefore not be unusual to find limitations here. In the remaining section, five possible limitations and their impact on this study will be discussed.

First, it might be inappropriate to generalise the findings of Study I to the whole Year 8 student population in Singapore in spite of a sizable number of students involved in this study. This is because their attainment profile was not typical of the entire student population. The median PSLE aggregate scores for the Express and Normal (Academic) student samples range from 222 to 234 and from 180 to 187 respectively whereas those for the cohort range from 200 to 285 for Express and 150 to 195 for Normal (Academic). However, this study did examine a group of 337 Express and 178 Normal (Academic) students from three secondary schools of fairly comparable academic backgrounds (see Table 3.5). With a wide spread of learning abilities amongst the students within each course (see Tables 3.8 and 3.9), it therefore appears reasonable to infer that the spread of students within each course in this study has little effect on the findings.

Using very much the same argument, the survey findings in Study II might not be generalised to the Singapore secondary school student population as well. As for the student interviews, their purpose was to get a sense of the reliability of the survey findings rather than generalisability of the outcomes. Only a small sample of eight Express and eight

Normal (Academic) students could be interviewed because of time and manpower constraints.

Next, given that the participating students came from three different schools and were taught by different mathematics teachers, it is possible that different teaching styles might have influenced the types of generalising strategies used, and consequently the types of rules produced as well as the types of justification schemes employed. However, the initial data analyses done separately for all the three groups of students did not seemingly display any marked differences in the types of rules, generalising strategies and justification schemes amongst the three groups. There was hardly any category of generalising strategies or justification schemes that was peculiar to any particular group because most were manifested by all the groups. As a result, the different teaching styles did not appear to have any significant bearing on the findings of Study I.

The use of three or even four successive configurations in figural generalising tasks is a very common approach in pattern generalisation research. These numbers are deemed sufficient for students to examine and compare in order to identify a pattern. So when a single or double configurations are presented in a generalising task instead of the usual numbers, it should come as no surprise that such a non-conforming approach raises concerns over the possibility of discerning a pattern from just one or even two configurations. However, generalising tasks with one or two configurations are not new and have appeared in a number of research studies, including, for instance, the well-recognised TIMSS. Additionally, the analysis of Express students' successes in the four *JuStraGen* tasks involving one or two configurations shows that the success rates were all 50% and above. Hence, providing one or even two configurations does appear to be adequate to allow students to detect the underlying pattern and then construct a rule.

Four different methods are offered in each *QBBS* task and these include one numerical and three figural approaches. Inevitably, concerns may arise over the small number of methods and the proclivity of the approaches towards the figural types. First, it is impractical to encompass all possible methods of developing a functional rule in each task because providing too many alternatives can cloud the students' thinking and stress them out especially when they cannot decide which method to choose from. Next, the decision to include more figural approaches has two considerations: (1) the solutions provided by in-service teachers during the pre-development stage, and (2) the call by some researchers to link the visual representation of a functional rule with its symbolic representation. It therefore appears reasonable to assume that the number and choice of methods are appropriate and will not markedly affect the results of Study II.

Finally, the construction of the analytic scoring rubric and the three coding schemes described under Section 3.4 relied on not only *a priori* ideas drawn from different sources but also the researcher's experience as a former secondary school mathematics teacher and a current mathematics educator, especially when creating new codes. The rubric and coding schemes can thus be biased in nature, and perhaps even open to numerous interpretations. In turn, the scoring and coding of student responses in the *JuStraGen* test can also result in biases as both rely on the researcher's as well as the two validators' interpretation of student responses and experience in teaching pattern generalisation. As a result, it hardly comes as a surprise that there were still some dissenting opinions over the scoring and coding even after training was provided to the validators. However, a 90% agreement level for scoring and a 94% agreement level for coding amongst the three validators seem adequate enough for the purpose of this study. Therefore, it seems reasonable to assume that the scoring rubric and coding schemes will not substantively affect the conclusion drawn.

6.4 IMPLICATIONS FOR TEACHING

The present study suggests two reasons why students were unsuccessful in expressing generality in their responses to the *JuStraGen* test. First, some just did not know what to focus on in their search for a pattern. Others examined the term-to-term relationship to develop a recursive rule rather than on the term-to-position relationship to establish a functional rule. To help both groups of students succeed, teachers might usefully begin a presentation around a generalising task by spending time exploring and discussing the pattern seen in the task. Students could be encouraged to articulate what they see in the pattern, and then asked to extend the pattern by predicting some terms that are both near to and far from the last given term in the pattern. The computation of these terms is a remarkably helpful exercise for those who do not immediately recognise any pattern structure. This was evidenced in student interviews when some students began to spot the pattern structure only after they were guided to find some near and far terms. From a pedagogic viewpoint, computing near and far terms is useful in helping students who have only articulated a recursive relationship come to two key realisations: (1) determining a near term requires them to know the term immediately preceding it and the differences between consecutive terms, and (2) the recursive approach is not an adequate method for determining a far term because the term immediately preceding it may not be available. Crucially, central to all these computations is to support students in recognising a need to devise a general rule to predict the far term directly.

Evidence from student justifications and interviews has illustrated that successful students tended to examine the term-to-position relationship in order to develop a functional rule. Thus to demonstrate the formulation of a functional rule, teachers could usefully emphasise the functional relationship between the input (i.e., the size number) and the output (i.e., the number of tiles or cards) variables from the outset. For instance, in *Bricks*, there are five bricks in Size 1, eight bricks in Size 2, and 11 bricks in Size 3. Once this is done, it is important for teachers to make clear to students about how the size number is being used as a generator of the relationship to connect it with its corresponding number of bricks. Elucidating this generator-term relationship is particularly crucial for developing a functional rule, so students should be encouraged to decide how the size number is linked to the term. This relationship could usefully be articulated verbally or written in words before expressing it algebraically. However, teachers must realise that the complexity of the students' verbal or written descriptions of the relationship may not easily transform into algebraically useful expressions and that their abilities to articulate a verbal or written description do not guarantee a successful translation of the relationship into its symbolic form.

As the results of this research have demonstrated, the same pattern structure can be envisioned using different generalising strategies. Of the various strategies, it is not uncommon for teachers to use their favourite one to illustrate how a pattern can be discerned. However, as the *QBBS* survey findings have shown, students have different preferences of generalising strategies. What this implies is that teachers should demonstrate several different approaches in addition to their own preferred choice. It is clear that not all students will necessarily be able to follow the teachers' favourite approach. Therefore, exposing students to other approaches will be beneficial as they would be able to pick the one that they feel able to understand and apply.

Study I showed that numerical strategies were commonly employed to work out the functional rules of figural patterns. What teachers must realise is that although certain strategies such as using the common difference and the zeroth term, and using the formula $a + (n - 1)d$, permit students to obtain a functional rule quickly, they do not necessarily encourage the students to inspect the position-to-term relationship and then abstract the pattern structure perceived into an expression. Some teachers may, however, argue that the numerical approach is versatile and very useful in solving both numerical and figural generalising tasks. There is nothing pedagogically wrong if they choose to teach such an approach, after all a correct functional rule can still be derived if the approach is carried out properly. But the important point here is that they need to be aware and mindful of its limitations in figural tasks. For instance, despite developing a functional rule, the numerical methods of generalising the pattern mentioned above might not foster algebraic thinking (Radford, 2008). Next, a student interview in Study II has highlighted one serious drawback with the numerical approach when visual cues are not drawn from the structure of configurations to construct the rule. By transforming the given figural pattern into a numerical sequence, a disconnection between the visual form of the pattern and its symbolic form can occur. When this happens, two totally different patterns can be produced, with the numerical one being incorrect.

The present study found compelling evidence of students favouring figural strategies in both the test and the survey, and also demonstrated the usefulness of such strategies over numerical ones in establishing the rules of quadratic patterns that do not conform to the typical square and triangle numbers. This suggests that when teaching figural tasks, teachers could encourage the use of figural generalising strategies. When teachers illustrate how a figural strategy can be applied to develop a rule directly from the configurations depicting the figural pattern, the linking of the visual form of the pattern to its symbolic form offers an explanation of the meaning to the rule. Thus making a conscientious attempt to explain the meaning can enhance the students' appreciation of the symbolic rule.

Amongst the numerous samples of student responses to the *JuStraGen* test, it was reported that few students who had established a correct functional rule symbolically introduced the alphabetical letter used to denote the size number. In the student interviews, some students were found to misunderstand the actual meaning of the letter. It is therefore important that teachers spend time describing what the letter used in the rule represents and encourage their students to do the same.

Evidence from this research highlighted students' inexperience with making justifications. With justification attracting considerable attention globally in recent years and a greater emphasis being placed on mathematical reasoning and communication, teachers then play an important role in stimulating students to engage in the justification process so as a step to helping them create a good justification. Teachers can begin by assuring students that making a good justification is not necessarily a difficult and daunting task. For students to understand what needs to be included in a justification, it is useful for them to be shown some examples to discuss and, whilst doing so, think about the purpose of the justification. For instance, in the case of justifying a functional rule generated in a pattern generalising task, a rational intent of the justification might be to make a student's thinking and reasoning visible to the teacher through an explanation of how the rule is constructed. Another sensible intent might be to convince the teacher that the rule does not occur by chance but it exists through its consistent modelling of a particular pattern. So it would be helpful if teachers explain to students that the justification is actually asking for an account of *how* the rule is developed. The description of this account can be provided entirely in words or in a combination of words, diagrams and symbols. With a few illustrations of the different ways to present a justification, teachers would be helping their students to grasp the requirements of justification.

6.5 IMPLICATIONS FOR TEACHER EDUCATION

The present study offers profound insights into how students think and reason about patterns. Given that teachers' knowledge of content shapes the way they teach, the wealth of rich data that emerged from this study can be used to broaden their content and pedagogical knowledge.

Several types of generalising strategies were identified in this study and teacher education should expose teachers to these strategies, in particular the figural kind, which most teachers might be ignorant of. Next, teachers seeking an idea of what might be an appropriate generalising strategy to employ in class when demonstrating examples should benefit from the findings of students' use of generalising strategies in Study I and beliefs of best-help strategies in Study II to help them make informed decisions, rather than simply relying on their beliefs about what students are capable of understanding and how best they will learn (Chua & Hoyles, 2010b). Getting teachers to align their choices of generalising strategies with the students' preferences will enhance students' learning of number patterns.

Given the multiple ways of seeing the structure of a pattern which then lead to constructing different-looking but equivalent rules, teacher education should also guide teachers to capitalise on this collection of equivalent rules to design meaningful teaching activities. For instance, a teaching activity they can develop involves getting their students to justify how each rule is equivalent to one another. Another teaching activity is to provide a rule and then invite students to explain how they think the rule is developed.

The present study has noticed that some students produced weak justifications that expose their ignorance of the justification requirements. What this implies is that teacher education should encourage teachers to think about what a reasonable justification should comprise. For instance, to provide an account of how a functional rule is developed, a good justification should indicate clearly the steps that lead to the abstraction of the pattern, illuminating the kind of generalising strategy used simultaneously with the help of diagrams where possible. Increased clarity of their own expectations will then empower them to articulate these expectations clearly in the teaching process. In expressing these expectations, they can also better scaffold their students to produce justifications that meet their requirements.

6.6 IMPLICATIONS FOR FUTURE RESEARCH

There is a wealth of pattern generalisation research studies in the literature and what the present study has hoped to achieve is to pave a path for future studies by providing some preliminary insights into the effect of two task features on students' generalisations and justifications, as well as into the generalising strategies, justification schemes and beliefs of Singapore secondary school students. Certainly, these worthwhile and meaningful areas deserve further investigations, and so, some suggestions for future research in these areas are presented in the subsequent paragraphs.

Since all the students in the present study come from the same age group in the Express and Normal (Academic) courses, future studies may involve students from other age groups in these courses so that the types of rules formulated and generalising strategies used can be examined and compared across the different age groups. Similar comparative studies investigating the justifications made by students and the kind of generalising strategy judged as most helpful by students from the different age groups are also worth analysing. In very much the same way, future studies may also be implemented in other countries in Asia and in the west so that the performance and beliefs of students internationally can be compared.

Since the *JuStraGen* test contains generalising tasks featuring three configurations, either successive or non-successive but all are arranged in ascending order, what will the students' rules, their generalising strategies and justification schemes be when the array of configurations change from ascending to "random" order? Will they still achieve the same success, formulate a variety of functional rules and use the same kind of generalising strategies and justification schemes? These areas are worth investigating. Thus another suggestion for future research is to conduct a study to examine the effect of the order of configurations on students' generalisations and justifications.

Given that the non-successive pattern format version of *JuStraGen* test contains two generalising tasks with each providing only one configuration accompanied by a brief general description of the configuration, it is possible that some researchers will argue that the students' ability to derive a functional rule is clearly assisted by the given description.

Will students still be able to discern a pattern and succeed in rule construction if the brief description is now omitted? This is another noteworthy area for future research.

Although the present study explores the influence of pattern format involving one to three non-successive configurations, it does not however investigate systematically the effect of the number of non-successive configurations on students' generalisations and justifications. Is a task with a single configuration more challenging than another task with double or even triple configurations? Thus another area worth investigating in the future is to conduct a study to examine whether or not the number of non-successive configurations influences students' generalising ability, the types of rules established, and the types of generalising strategies and justification schemes used.

6.7 CONCLUDING REMARKS

The present study is designed to explore the generalisations, justifications and beliefs of Secondary Two students in Singapore, and to investigate the effects of the format of pattern display and the type of functions on the first two aspects. The study has shed some light on these areas. Several findings of this study, together with the implications for teaching and teacher education, as well as suggestions for future research studies, have already been reported and discussed in great detail above. The conclusions that emerge from an examination of the findings of this study are that a significant number of academically more able students performed competently in linear generalising tasks, but faltered in quadratic generalising tasks whereas a vast majority of the less academic students succumbed completely in both linear and quadratic generalising tasks. As a result, the common impression that expressing generality is elusive appears to be confirmed in this study for Singapore students. Several generalising strategies and justification schemes that were popular amongst the students were also identified. The greatest strength of the present study lies in the discovery that students' difficulties in generalisations were influenced by the format of pattern display and the type of functions. It is therefore hoped that with a greater awareness of the types of generalising strategies and justification schemes that students prefer to use, and the factors that can contribute to their learning difficulties in making generalisations, teachers can then subsequently plan more effective teaching and learning to help students construct and justify their rules underpinning linear or quadratic patterns. Finally, the field of pattern generalisation, being an extensively well-researched area, is definitely still worthy of further investigations in an effort to improve students' learning of number patterns and teachers' pedagogical knowledge of number patterns.

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APPENDIX 1: LETTER OF CONSENT AND INFORMATION SHEET FOR SCHOOL

July 2011

RESEARCHER'S COPY

Please sign and return to school

Mr X
Head of Mathematics Department
XX Secondary School

Dear Mr X,

I am currently conducting a study looking at how Singapore secondary school students express mathematical generalisation of number patterns. To obtain a representative sample, I am approaching a number of schools to collaborate in this effort. I am hoping you and your students may be interested in participating.

Each student will be asked to answer a number of pattern tasks in the form of a written test. The pattern in each task is presented in a pictorial context from which students will need to find a general rule to describe the pattern. The pattern tasks are divided into two tests. Each student will sit the two tests that are administered on separate days, each lasting about an hour.

Subsequently, each student will be asked to answer a few questions in the form of a questionnaire. Adapted from the pattern tasks in the tests above, each question presents four different methods of finding a general rule that underpins the same pattern. Students will have to pick the method that they judge as most helpful in finding the general rule. The questions are also divided into two sets. Each set will be administered immediately after each written test. Each student will take the two surveys, each lasting 20 minutes.

Some students will then be selected based on their answers given in the tests, as well as surveys, and interviewed. Because the tests, surveys and interviews will be conducted after school, there will be minimal disruption to class lessons. The privacy of students' data will be protected and results from each school will be treated as confidential.

Results from this study will provide information on the kind of (a) generalising strategies that students used to derive a general rule, (b) generalising strategies that students judge as most helpful in deriving a general rule, and (c) justifications that students give for illustrating how they detect the pattern and derive the general rule. This information may support teachers in designing effective lessons that will assist in students' learning. Furthermore, the findings may provide teachers with knowledge to support their classroom interactions with students when they are engaged in making generalisations.

Although your child's participation is both highly appreciated and is of vital importance to this study, participation is voluntary. If you wish to allow your students to participate, please complete the form printed on the other side of this letter. Please sign and date both your copy and the researcher's copy. Return the researcher's copy to me. Keep the other copy for your own reference.

If you require further information, you can call me at 6790 3971 or email me at boonliang.chua@nie.edu.sg.

Thank you very much.

Chua Boon Liang

PhD Candidate

Institute of Education, University of London

I agree to allow my students to participate in the study described overleaf. I have read and understood the requirements of the study. Furthermore, I understand that (a) participation is voluntary, (b) both my students and I have the right to terminate participation at any time, and (c) both my students and I have the right to have collected data treated in a secured and confidential manner.

______________________________	_________________	______________
Your name (in BLOCK letters)	Signature	Date

______________________________	_______________________________________
Your designation	Name of school

APPENDIX 2(A): GENERALISATION ATTAINMENT TEST

Participant's Details

Name:		Class:	Sec 2 _______
Gender:	Male / Female *(please circle)*	Stream:	Express / Normal (Academic) *(please circle)*

Instructions for Participants

(a) This question paper consists of **8** printed pages.
(b) Answer all questions.
(c) Show your working clearly as marks may be awarded for correct working.
(d) Write your answers in this booklet.
(e) You are given **1 hour** to complete the questions.
(f) Calculators may be used.
(g) ***Your results and identities will be kept strictly confidential.***

For Researcher's Use Only

Student Code:		**School Code:**	

	Question Number	**Marks**
MCQ	1 – 10	/ 20
Short-answer questions	11 – 16	/ 6
	17 – 21	/ 10
	22 – 23	/ 6
Structured questions	24 – 25	/ 8
	Total Marks	/ 50

Objectives	**Question Number**	**Marks**
1	4, 6, 11, 18	/7
2	1, 7, 12, 16, 19	/8
3a	3, 14, 17, 24(a)	/7
3b	9, 24(b), 25(b)	/6
4a	2, 23	/5
4b	8, 13, 20, 24	/8
5	5, 9, 15, 21, 25(a)	/9
	Total Marks	/ 50

PART 1: Questions 1 to 10 carry 2 marks each.

For each question, five options (A, B, C, D or E) are given. One of them is the correct answer.

Make your choice and write down the letter (A, B, C, D or E) representing your choice in the brackets shown at the end of the question. (20 marks)

---

1. $\mathrm{mn} - 2\mathrm{mn} + 3\mathrm{mn} + 6\mathrm{mn}$, when simplified, is equal to

(A) 8
(B) $8\mathrm{mn}$
(C) $9\mathrm{mn} - 2$
(D) $9\mathrm{m}^2\mathrm{n}^2 - 1$
(E) $10\mathrm{mn}$ **( )**

2. The n^{th} term of a number sequence is $2n + 3$.

Which *one* of the following sequences shows the first five terms of this number sequence?

(A) 2, 5, 8, 11, 14, …
(B) 3, 5, 7, 9, 11, …
(C) 3, 6, 9, 12, 15, …
(D) 5, 7, 9, 11, 13, …
(E) 5, 8, 11, 14, 17, … **( )**

3. Given that $\mathrm{S} = 4\mathrm{r}^2$, the value of S when $\mathrm{r} = 3$ is

(A) 13
(B) 24
(C) 36
(D) 49
(E) 144 **( )**

4. A rectangle has length p cm and breadth $p - 2$ cm. The perimeter, in cm, is

(A) $2(p - 1)$
(B) $2(2p - 1)$
(C) $4p - 1$
(D) $4(p - 1)$
(E) $p(p - 2)$ **( )**

5. Peter used a rule to obtain the following first five terms of a number sequence:

14, 11, 8, 5, 2, …

Which *one* of the following rules for the n^{th} term of the number sequence could Peter have used to obtain the five terms above?

(A) $n - 3$
(B) $3n - 17$
(C) $14 - n$
(D) $14 - 3n$
(E) $17 - 3n$ **( )**

6. An empty jar weighs x grams.

It can hold 30 grams of honey.

The weight, in grams, of 12 full jars of honey is

(A) $12x + 30$
(B) $12x + 360$
(C) $30x + 12$
(D) $360x$
(E) $360 + x$ **( )**

7. Which *one* of the following algebraic expressions does **NOT** simplify to the same answer?

(A) $1 + 3(n-1)$
(B) $2(2n-1)-n$
(C) $2(n-2) + (n-1)$
(D) $(2n-1) + (n-1)$
(E) $n(2n-1)-2(n-1)^2$ **( )**

8. Given the number sequence 1, 4, 7, 10, 13, …, which *one* of the following numbers is also a term of this sequence?

(A) 28
(B) 33
(C) 47
(D) 54
(E) 60 **( )**

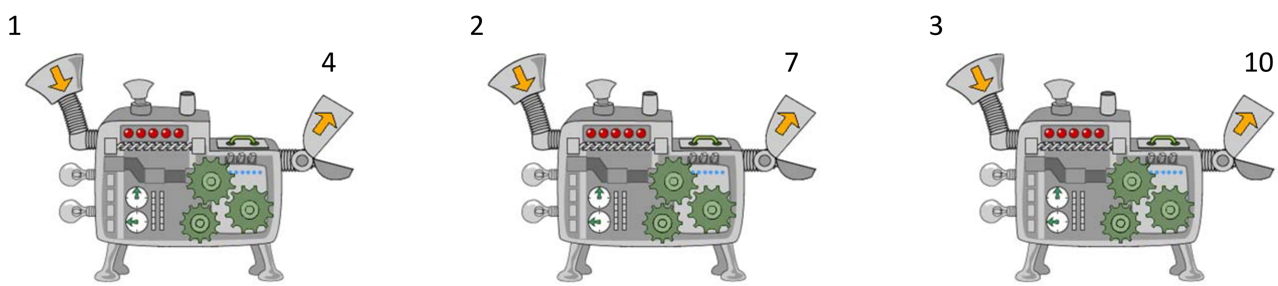
9. Consider the ordered pairs: (3, 12), (6, 21), and (8, 27).

Which *one* of the following rules describes how to get the second number from the first number in every ordered pair above?

(A) Add 1 and then multiply by 3.
(B) Add 9.
(C) Multiply by 2 and then add 6.
(D) Multiply by 4.
(E) Subtract 1 and then multiply by 6. **( )**

10. Kumar has a toy machine.

When he puts the number 1 into the machine, it produces the number 4. When the number 2 is put into this machine, it produces the number 7. And when the number 3 is put into it, the number 10 is produced.



Kumar puts a number into this machine but forgets to look at this number. The machine produces the number 40.

Which *one* of the following numbers could Kumar have put into the machine?

(A) 9
(B) 10
(C) 11
(D) 12
(E) 13 **( )**

PART 2: Questions 11 to 16 carry 1 mark each.
Questions 17 to 21 carry 2 marks each.
Questions 22 and 23 carry 3 marks each.
Questions 24 and 25 carry 4 marks each.

Show your working clearly and write your answers in the space provided. (30 marks)

---

11. Express the statement below as an algebraic expression.

Subtract 7 from twice of x.

Answer: ________________ [1]

12. Simplify $3m + 6m + k$.

Answer: ________________ [1]

13. Write down the next term of the number sequence: 23, 20, 17, 14, … .

Answer: ________________ [1]

14. Given the formula $\mathrm{T} = \frac{1}{2}\mathrm{n}(\mathrm{n}+1)$, find the value of T when $\mathrm{n} = 20$.

Answer: ________________ [1]

15. The first four terms of a number sequence are 3, 4, 5, 6, …
Write down an expression for finding the n^{th} term of this number sequence.

Answer: ________________ [1]

16. Simplify $5n - 9 - 3n + 4$.

Answer: ________________ [1]

17. Lily is finding values of y for different values of x in the formula $y = 2x - 3$.
She makes a table but gets one value of y wrong.

x	2	5	7	11	20
y	1	7	11	19	35

Circle the wrong value of y in the table and write down its correct value.

Answer: ________________ [2]

18. Express the statement below as an algebraic expression.

Multiply the square of y by 5 and then add to 3 times y.

Answer: ________________ [2]

19. Simplify $13 + 3(n - 5)$.

Answer: ________________ [2]

20. Fill in the **two** missing terms in the following sequence: [2]

23, ____, 39, 47, ____, 63, …

21. The first four terms of a number sequence are 1×2, 2×3, 3×4, 4×5, …

Write down an expression for finding the n^{th} term of this number sequence.

Answer: ____________________ [2]

22. The n^{th} term of a number sequence is $7 - 3n$.

Write down the **first three** terms of this sequence.

Answer: ____________________ [3]

23. Fill in the **three** missing terms in the following sequence: [3]

68, 59, 51, ____, 38, ____, 29, 26, ____, …

24. The rule that Mary uses for getting a number sequence is given below.

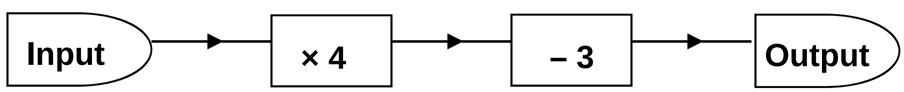
Add 1 to the previous term and then multiply by 2

(a) If Mary starts with the number 3, find the third number in the sequence.
(b) If Mary begins with a different starting number, the second number that she obtains is 16. What is the starting number Mary has used?

Answer: (a) ______________ [2]

(b) ______________ [2]

25. John uses the following rule to obtain the values in the table below:



Input	Output
1	1
2	5
3	9
4	13

(a) What is the output when the input is n?
(b) Find the input value that John will have to use so as to obtain the output value of 77.

Answer: (a) ______________ [2]

(b) ______________ [2]

END OF PAPER

APPENDIX 2(B): ANSWER KEY FOR GENERALISATION ATTAINMENT TEST

Question Number	Answers	Marks
1	B	2 marks each
2	D	
3	C	
4	D	
5	E	
6	B	
7	C	
8	A	
9	A	
10	E	(Total: 20 marks)
11	$2x - 7$	1 mark each
12	$9m + k$	
13	11	
14	210	
15	$3 + (n - 1)$ or $n + 2$	
16	$2n - 5$	(Total: 6 marks)
17	Circle wrong value 35 Correct value: 37	1 mark 1 mark (Total: 2 marks)
18	$5y^2$ or $3y$ seen $5y^2 + 3y$	1 mark 1 mark (Total: 2 marks)
19	$3n - 15$ $3n - 2$	1 mark 1 mark (Total: 2 marks)
20	31 55	1 mark 1 mark (Total: 2 marks)
21	$(n + 1)$ seen $n(n + 1)$ or $n \times (n + 1)$	1 mark 1 mark (Total: 2 marks)

22	4, 1, – 2	1 mark each (Total: 3 marks)
23	44, 33, 24	1 mark each (Total: 3 marks)
24	(a) Second term 8 seen Third term 18 seen (b) $\frac{16}{2}$ or 8 seen $8 - 1 = 7$	1 mark 1 mark 1 mark 1 mark (Total: 4 marks)
25	(a) $4n$ seen $4n - 3$ (b) $4n - 3 = 77$ $4n = 80$ or $77 + 3 = 80$ $n = 20$ or $\frac{80}{4} = 20$	1 mark 1 mark 1 mark 1 mark (Total: 4 marks)

APPENDIX 3(A): JUSTRAGEN TEST SET 1S

Participant's Details

Name:		Class:	Sec 2 _______
Gender:	Male / Female *(please circle)*	Stream:	Express/Normal (Academic) *(please circle)*

Instructions for Participants

(h) This question paper consists of **5** printed pages.
(i) Answer four (**4**) questions in this test.
(j) Show your answer and working clearly in the answer space provided below each task.
(k) You are given **45 minutes** to complete the questions.
(l) Calculators may be used.
(m) ***Your results and identities will be kept strictly confidential.***

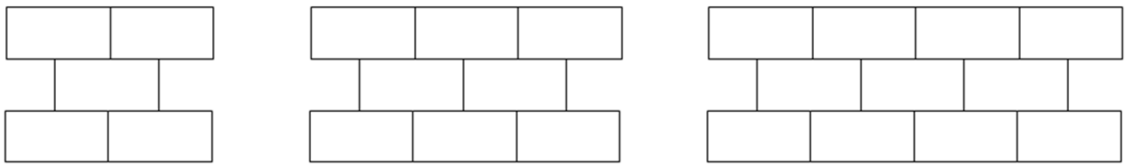
For Researcher's Use Only

Student Code:		
Q1: Bricks		
Q2: Oh Deer!		
Q3: Birthday Party Decorations		
Q4: Tulips		

Question 1: Bricks

John used identical bricks to make several designs of different sizes on a long wall.

The diagrams below show three designs he made.



Size 1 Size 2 Size 3

As the size number became larger, more bricks were used.
John wanted to find the number of bricks he had to use to make any size.
He used a rule to find this number.

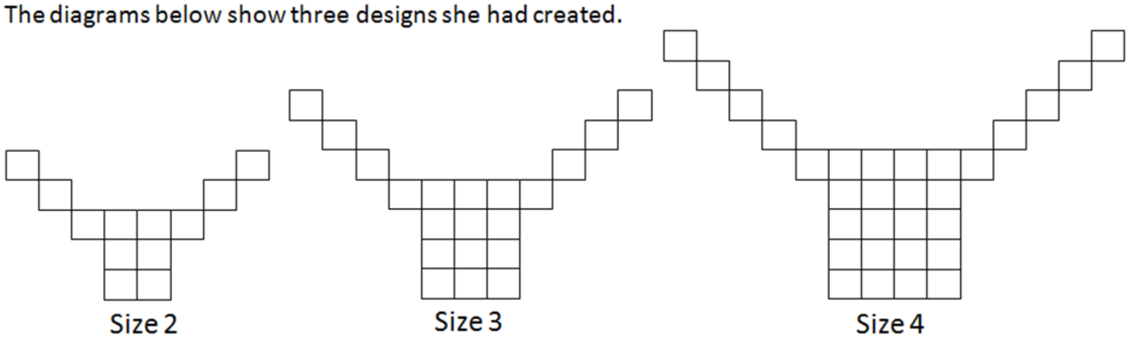
(a) Write down the rule John might have used in terms of the size number.

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Question 2: Oh Deer!

Sally used identical square cards to create designs of deer head in different sizes.

The diagrams below show three designs she had created.



As the size number became larger, more square cards were used.
Sally wanted to find the number of square cards she had to use to create designs of any sizes.
She used a rule to find this number.

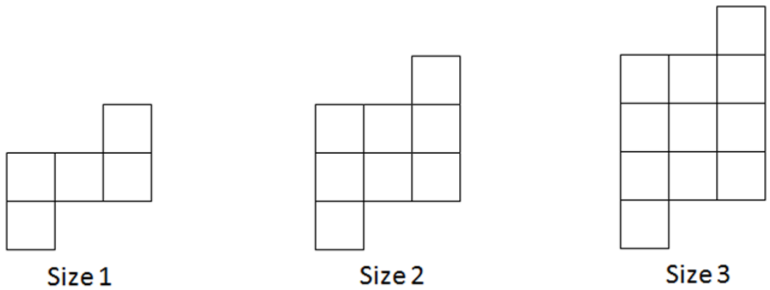
(a) Write down the rule Sally might have used in terms of the size number.

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Question 3: Birthday Party Decorations

Mary used identical square cards to make several birthday party decorations of different sizes.

The diagrams below show three party decorations she made.



As the size number became larger, more square cards were used.
Mary wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

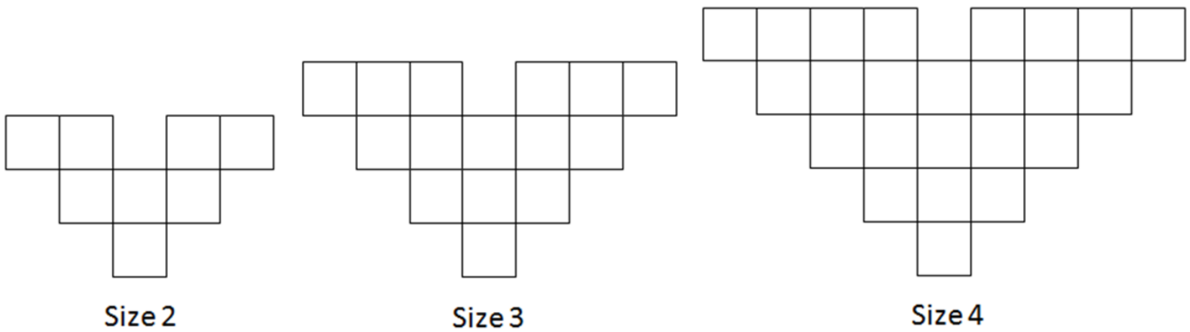
(a) Write down the rule Mary might have used in terms of the size number.

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Question 4: Tulips

Tony used identical square tiles to create flower designs of different sizes for his art project.

The diagrams below show three flower designs he made.



As the size number became larger, more square tiles were used.
Tony wanted to find the number of square tiles he had to use to make any size.
He used a rule to find this number.

(a) Write down the rule Tony might have used in terms of the size number.

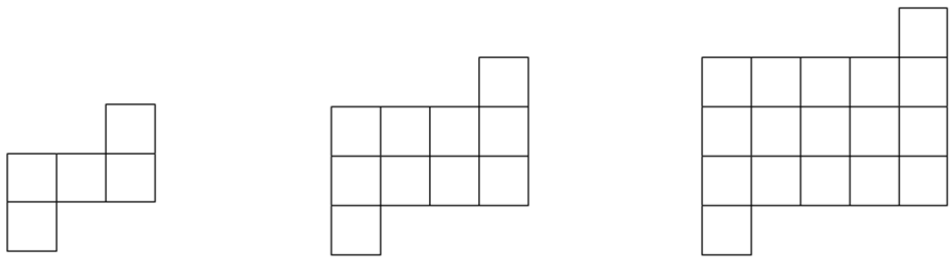
(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

APPENDIX 3(B): JUSTRAGEN TEST SET 2S

Question 1: Christmas Party Decorations

Alice used identical square cards to make several Christmas party decorations of different sizes.

The diagrams below show three party decorations she made.



Size 1 Size 2 Size 3

As the size number became larger, more square cards were used.
Alice wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

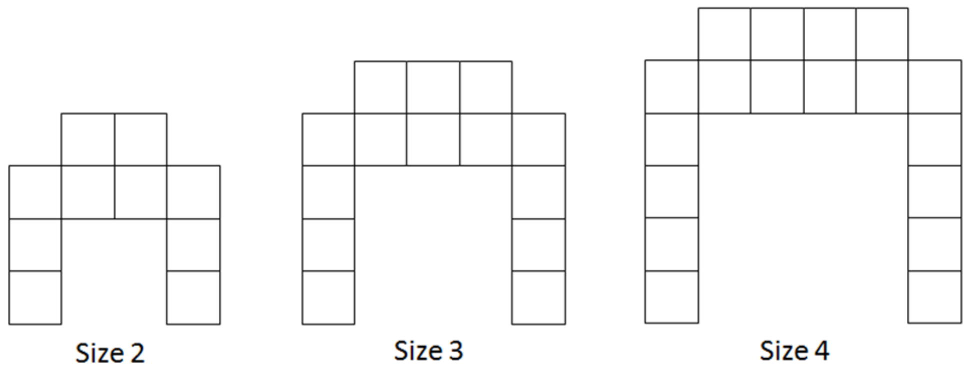
(c) Write down the rule Alice might have used in terms of the size number.

(d) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Question 2: Towers

Tom built towers of different sizes by using identical square tiles.

The diagrams below show three towers he had built.



As the size number became larger, more square tiles were used.
Tom wanted to find the number of square tiles he had to use to build towers of any sizes.
He used a rule to find this number.

(c) Write down the rule Tom might have used in terms of the size number.

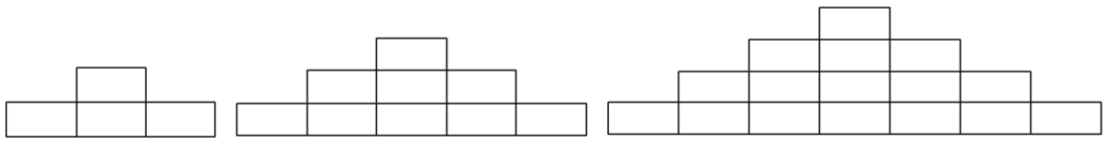
(d) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Question 3: Wall Design

Alan used identical bricks to make several designs of different sizes on a long wall.

Each design is a stack of bricks.
The topmost row contains only one brick.
Each row below has two more bricks than the row above it.

The diagrams below show three of the designs he made.



Size 1 Size 2 Size 3

As the size number became larger, more bricks were used.
Alan wanted to find the number of bricks he had to use to make any size.
He used a rule to find this number.

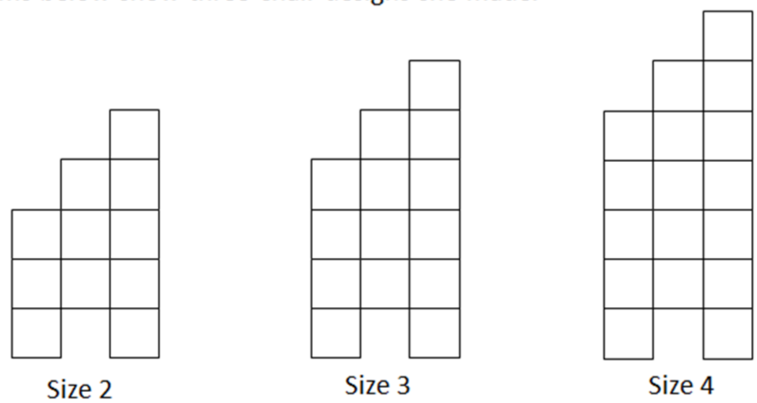
(c) Write down the rule Alan might have used in terms of the size number.

(d) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Question 4: High Chairs

Ruby used identical square cards to make chair designs of different sizes for her art project.

The diagrams below show three chair designs she made.



As the size number became larger, more square cards were used.
Ruby wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

(c) Write down the rule Ruby might have used in terms of the size number.

(d) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

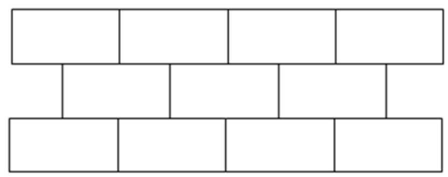
APPENDIX 3(C): JUSTRAGEN TEST SET 1NS

Question 1: Bricks

John used identical bricks to make several designs of different sizes on a long wall.

Each design is made up of three rows of bricks.
The top and bottom rows are identical, containing the same number of bricks.
The middle row is shorter and has one fewer brick than each of the other two rows.

The diagram below shows how a Size 3 design that John made looks like.



Size 3

As the size number became larger, more bricks were used.
John wanted to find the number of bricks he had to use to make any size.
He used a rule to find this number.

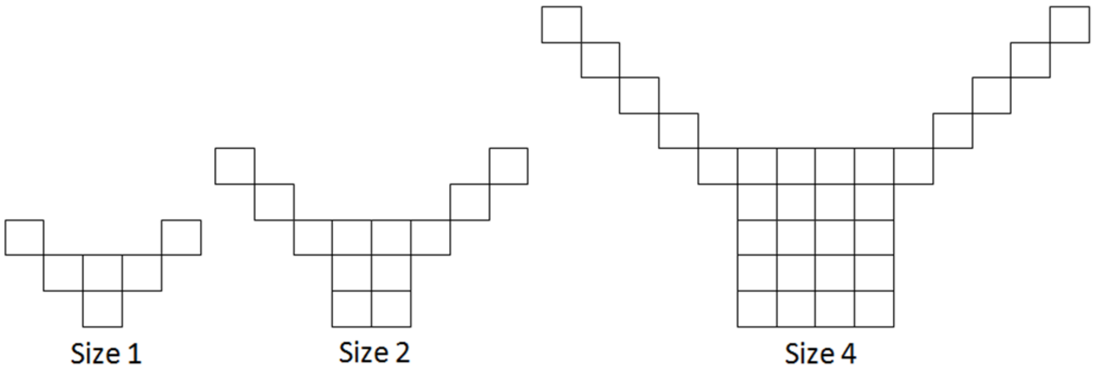
(e) Write down the rule John might have used in terms of the size number.

(f) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Question 2: Oh Deer!

Sally used identical square cards to create designs of deer head in different sizes.

The diagrams below show three designs she had created.



As the size number became larger, more square cards were used.
Sally wanted to find the number of square cards she had to use to create designs of any sizes.
She used a rule to find this number.

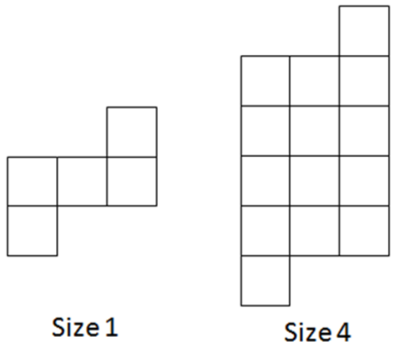
(e) Write down the rule Sally might have used in terms of the size number.

(f) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Question 3: Birthday Party Decorations

Mary used identical square cards to make several birthday party decorations of different sizes.

The diagrams below show two party decorations she made.



As the size number became larger, more square cards were used.
Mary wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

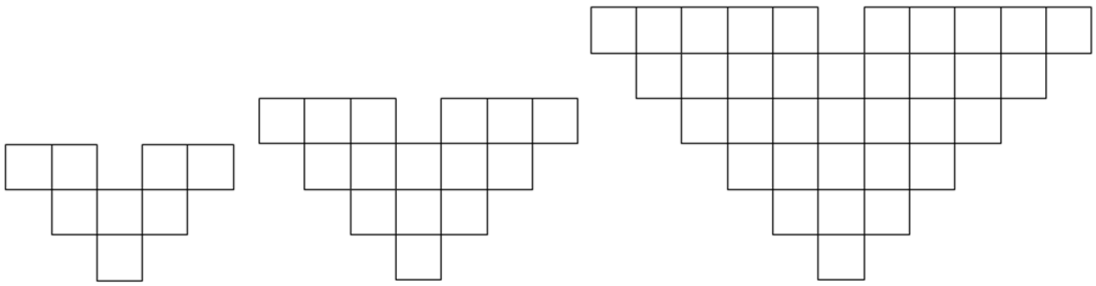
(e) Write down the rule Mary might have used in terms of the size number.

(f) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Question 4: Tulips

Tony used identical square tiles to create flower designs of different sizes for his art project.

The diagrams below show two flower designs he made.



Size 2 Size 3 Size 5

As the size number became larger, more square tiles were used.
Tony wanted to find the number of square tiles he had to use to make any size.
He used a rule to find this number.

(e) Write down the rule Tony might have used in terms of the size number.

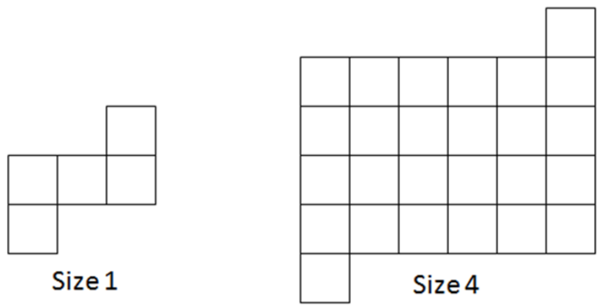
(f) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

APPENDIX 3(D): JUSTRAGEN TEST SET 2NS

Question 1: Christmas Party Decorations

Alice used identical square cards to make several Christmas party decorations of different sizes.

The diagrams below show two party decorations she made.



As the size number became larger, more square cards were used.
Alice wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

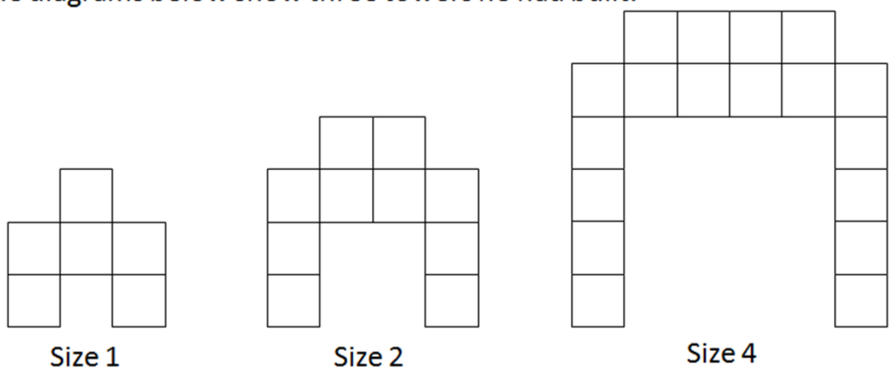
(g) Write down the rule Alice might have used in terms of the size number.

(h) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Question 2: Towers

Tom built towers of different sizes by using identical square tiles.

The diagrams below show three towers he had built.



As the size number became larger, more square tiles were used.
Tom wanted to find the number of square tiles he had to use to build towers of any sizes.
He used a rule to find this number.

(g) Write down the rule Tom might have used in terms of the size number.

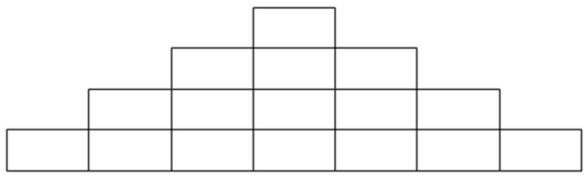
(h) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Question 3: Wall Design

Alan used identical bricks to make several designs of different sizes on a long wall.

Each design is a stack of bricks.
The topmost row contains only one brick.
Each row below has two more bricks than the row above it.

The diagram below shows how a Size 3 design that Alan made looks like.



Size 3

As the size number became larger, more bricks were used.
Alan wanted to find the number of bricks he had to use to make any size he had in mind.
He used a rule to find this number.

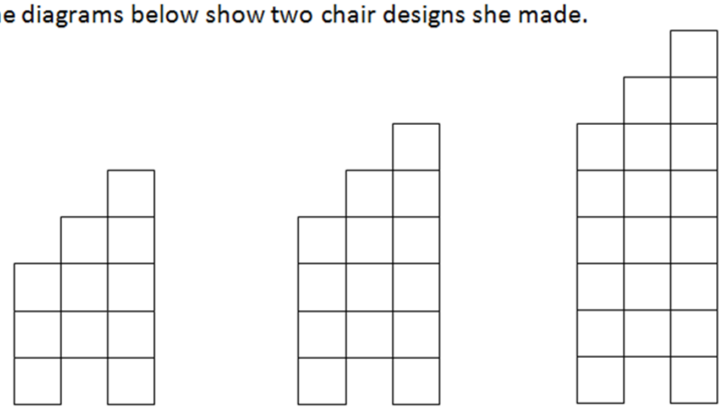
(g) Write down the rule Alan might have used in terms of the size number.

(h) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

Question 4: High Chairs

Ruby used identical square cards to make chair designs of different sizes for her art project.

The diagrams below show two chair designs she made.



Size 2 Size 3 Size 5

As the size number became larger, more square cards were used.
Ruby wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

(g) Write down the rule Ruby might have used in terms of the size number.

(h) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

APPENDIX 4(A): QBBS 1

Participant's Details

Name:		Class:	Sec 2 _______
Gender:	Male / Female *(please circle)*	Stream:	Express/Normal (Academic) *(please circle)*

Instructions for Participants

(a) This questionnaire consists of **5** printed pages.
(b) Answer **two** (**2**) questions in this questionnaire.
(c) The two questions were taken from the test that you just took earlier. Each question shows four different methods of working out the rule. Study the four methods carefully, then answer the questions that follow.
(d) You are given **15 minutes** to complete the questions.
(e) Calculators may be used.
(f) ***Your results and identities will be kept strictly confidential.***

For Researcher's Use Only

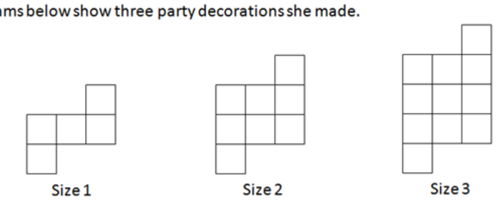
Student Code:		
Q1: Birthday Party Decorations		
Q2: Bricks		

Question 1: Birthday Party Decorations

Four different methods were used by a group of students (Anne, Ben, Clark and Dawn) to work out the general rule for the task on the right.

Mary used identical square cards to make several birthday party decorations of different sizes.

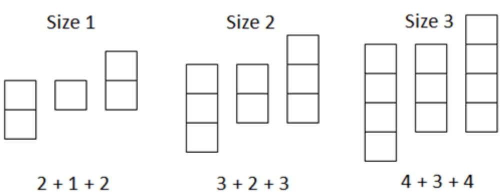
The diagrams below show three party decorations she made.



As the size number became larger, more square cards were used.
Mary wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

The following discussion took place amongst the four students.

Anne: That's easy. I got the rule by separating the decorations into three parts as follows.



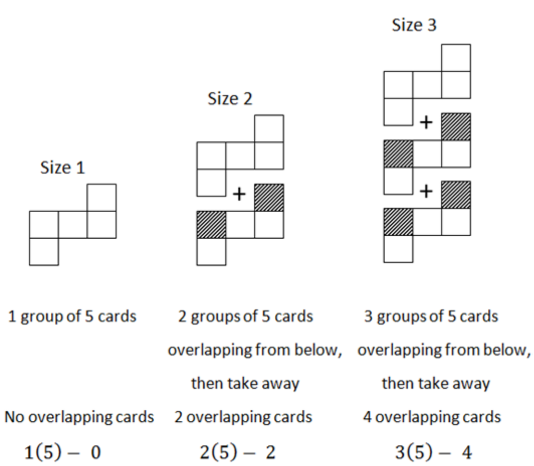
Method 1

Ben: Wait a minute. I counted the number of cards in each decoration, recorded them in a table, then obtained the rule from the table.

Size number	Number of square cards used	
1	5	5
2	8	8 = 5 + 3
3	11	11 = 5 + 3 + 3
:	:	:

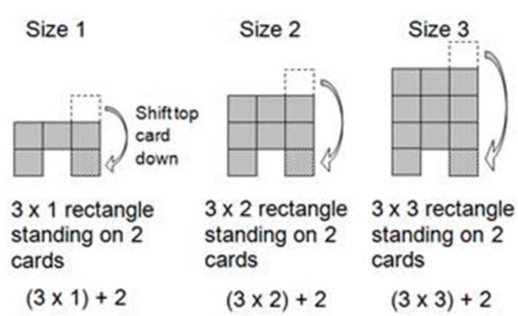
Method 2

Clark: For me, I figured out the rule by using overlapping of shapes.



Method 3

Dawn: Well, this is how I worked out the rule. I shifted the top card down to form a rectangle that stands on two cards.



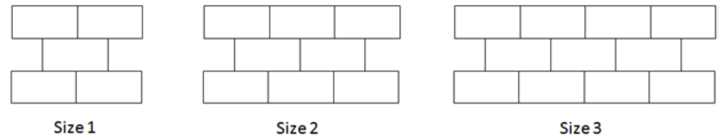
Method 4

Question 2: Bricks

Four different methods were used by a group of students (Anne, Ben, Clark and Dawn) to work out the general rule for the task on the right.

John used identical bricks to make several designs of different sizes on a long wall.

The diagrams below show three designs he made.



As the size number became larger, more bricks were used.
John wanted to find the number of bricks he had to use to make any size.
He used a rule to find this number.

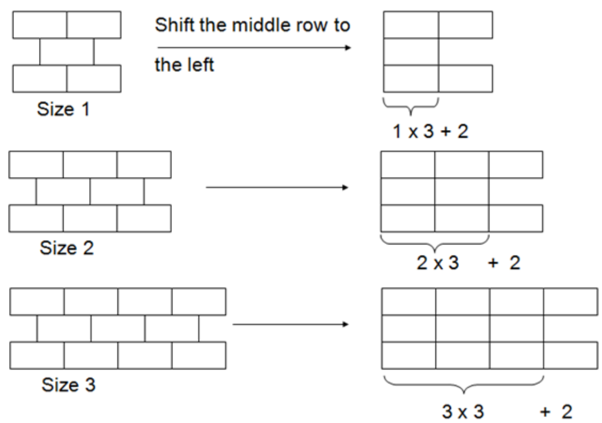
The following discussion took place amongst the four students.

Anne: I counted the number of bricks in each design, recorded them in a table, then worked out the rule from the table.

Size Number	No of bricks used	
1	5	5
2	8	8 = 5 + 3
3	11	11 = 5 + 3 + 3
:	:	:

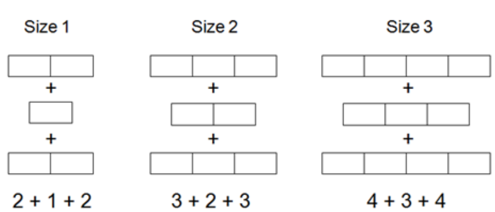
Method 1

Ben: Well, this is how I worked out the rule. I first shifted the middle row to the left. Then I see a pattern in the new shapes.



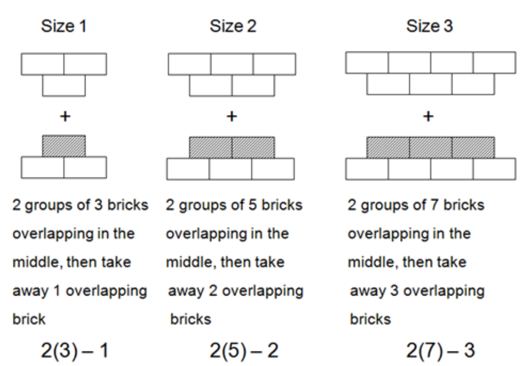
Method 2

Clark: That's easy. I got the rule by separating the designs into three parts as follows.



Method

Dawn: For me, I figured out the rule by using overlapping of shapes.



Method

For Question 1

Which method do you believe would best help **you** to work out the rule?

*Tick **only one***:	Method 1	Method 2	Method 3	Method 4
	☐	☐	☐	☐

For Question 2

Which method do you believe would best help **you** to work out the rule?

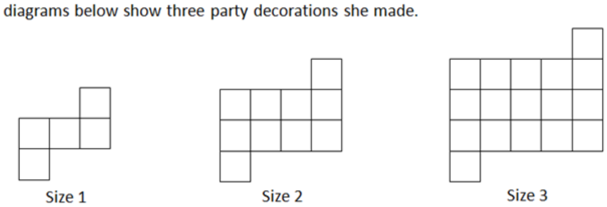
*Tick **only one***:	Method 1	Method 2	Method 3	Method 4
	☐	☐	☐	☐

Question 1: Christmas Party Decorations

Four different methods were used by a group of students (Anne, Ben, Clark and Dawn) to work out the general rule for the task on the right.

Alice used identical square cards to make several Christmas party decorations of different sizes.

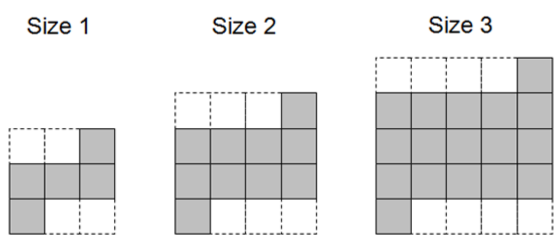
The diagrams below show three party decorations she made.



Alice wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

The following discussion took place amongst the four students.

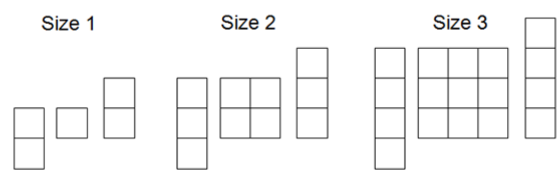
Anne: For me, I got the rule by first imagining the given decorations as part of a big square, then minus two rows of missing



$3^2 – (2 \times 2)$ $4^2 – (2 \times 3)$ $5^2 – (2 \times 4)$

Method 1

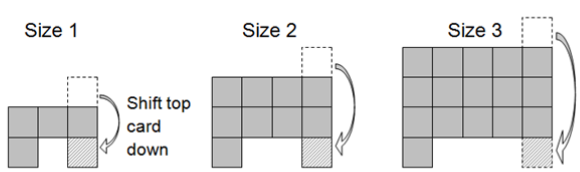
Ben: That's easy. I got the rule by separating the decorations into three parts as follows.



$2 + 1^2 + 2$ $3 + 2^2 + 3$ $4 + 3^2 + 4$

Method 2

Clark: Well, this is how I worked out the rule. I shifted the top card down to the last row to form a rectangle that stands on two cards.



3 x 1 rectangle on 2 cards — $(3 \times 1) + 2$

4 x 2 rectangle on 2 cards — $(4 \times 2) + 2$

5 x 3 rectangle on 2 cards — $(5 \times 3) + 2$

Method 3

Dawn: I counted the number of cards in each decoration, recorded them in a table, then worked out the rule from the table.

Size number	Number of square cards used	
1	5	5
2	10	10 = 5 + 5
3	17	17 = 5 + 5 + 7
4	26	26 = 5 + 5 + 7 + 9
:	:	:

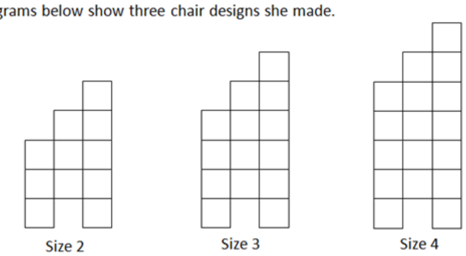
Method 4

Question 2: High Chairs

Four different methods were used by a group of students (Anne, Ben, Clark and Dawn) to work out the general rule for the task on the right.

Ruby used identical square cards to make chair designs of different sizes for her art project.

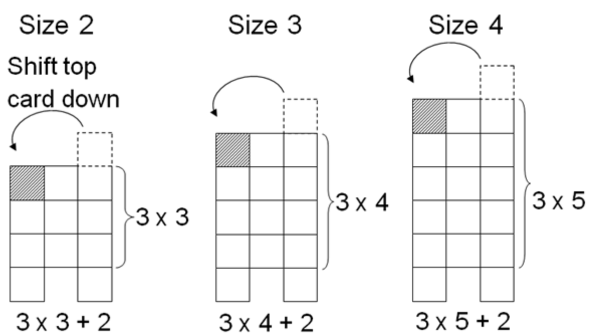
The diagrams below show three chair designs she made.



As the size number became larger, more square cards were used.
Ruby wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

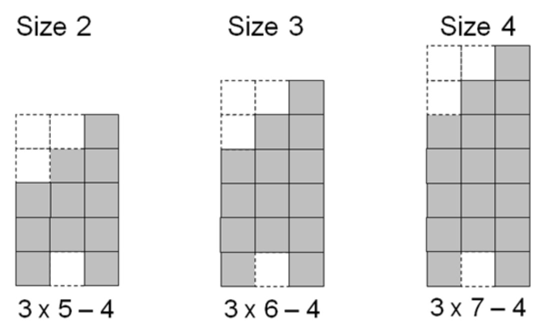
The following discussion took place amongst the four students.

Anne: Well, this is how I worked out the rule. I shifted the top card down to the next row to form a rectangle that stands on two cards.



Method 1

Ben: That's easy. I got the rule by first imagining the given designs as part of a big rectangle, then minus four cards.



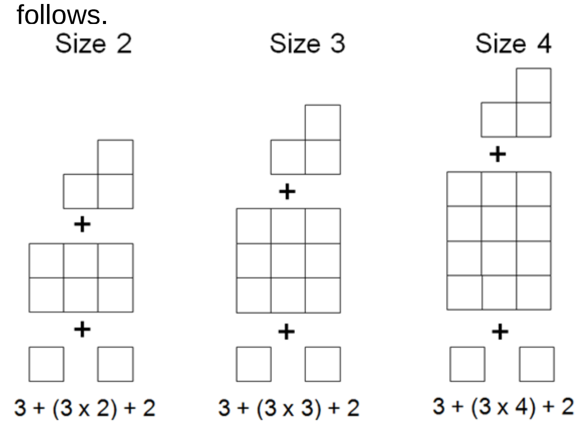
Method 2

Clark: I counted the number of cards in each design and recorded them in a table. Then I worked backwards to get Size 1. Finally I worked out the rule from the table.

Size Number	No of cards used	
1	8	8
2	11	11 = 8 + 3
3	14	14 = 8 + 3 + 3
4	17	17 = 8 + 3 + 3 + 3
:	:	:

Method 3

Dawn: For me, I figured out the rule by separating the designs into three parts as follows.



Method 4

For Question 1

Which method do you believe would best help **you** to work out the rule?

*Tick **only one***:	Method 1	Method 2	Method 3	Method 4
	☐	☐	☐	☐

For Question 2

Which method do you believe would best help **you** to work out the rule?

*Tick **only one***:	Method 1	Method 2	Method 3	Method 4
	☐	☐	☐	☐

APPENDIX 5: ANALYTIC SCORING RUBRIC FOR JUSTRAGEN TEST

Part 1: Rule construction

Score	Descriptors
5	A correct functional rule is given.
4	**(a)** An incorrect functional rule is given due to *minor errors*, *slips*. or *incorrect manipulation of symbols*. **(b)** A workable rule, not expressed in terms of the size number, is given.
3	**(a)** The components of the figure are correctly expressed in terms of the size number, but *not strung together* to produce a functional rule. **(b)** The numerical structures of (at least three) terms are correctly given, but the functional rule is *not stated*.
2	A correct recursive rule is given.
1	**(a)** First differences are correct and seen but the recursive rule is not explicitly stated. **(b)** The recursive rule is incorrectly expressed in symbols. **(c)** A correct description, in words or pictorially, is given to show how a particular shape is built or a particular term is obtained.
0	**(a)** Partially correct answer but not leading to any recursive or functional rule. **(b)** Wrong or no answer is presented.

Part 2: Use of Generalising Strategies

Score	Descriptors	Remarks
5	Evidence of appropriate use of numerical or visual cues established directly from the pattern to correctly derive a functional rule	Must be able to recognise the generalising strategy used
4	Evidence of an attempt to work out the *structure* of at least one particular non-immediate term, but there may not be any indication of generality or students are incapable of obtaining a correct functional rule	For example, if the greatest size number given is 4, the non-immediate term should be at least size number 6.
3	**(a)** Evidence of an attempt to work out the *structure* of at least one particular immediate term, but there may not be any indication of generality or students are incapable of obtaining a correct functional rule **(b)** Evidence of correct pattern-spotting	For example, if the greatest size number given is 4, the immediate term could be size number 3 or 5. Correct rule given – 5 points Strategy unclear – 3 points (lowest possible score is awarded)
2	Evidence of using the differences between consecutive terms to obtain a recursive rule	
1	Evidence of looking at the nature of the terms	(eg, Some terms are even, some are odd. Or multiple of 3, multiple of 4, multiple of 5)
0	**(a)** No evidence in the case of blank response. **(b)** Evidence of incorrect pattern-spotting. **(c)** Evidence of using an incorrect generalising strategy.	

APPENDIX 6(A): CODING SCHEME FOR RULE TYPES FOR BRICKS

SPSS Code	Rule generated for Q1 (Bricks)
101	$3n + 2$
102	$5 + 3(n - 1)$
103	$2(n + 1) + n$
104	$n + (2n + 2)$
105	$2n + (n + 2)$, $2n + (n + 1) + 1$
106	$3(n + 1) - 1$
107	$4n - (n - 2)$
108	$2(n + 2) + n - 2$
120	Recursive rule showing how the next term is obtained from the previous term (e.g., “Add 3” rule, increase the bricks by 3, or add 1 brick to the top, middle and bottom row)
950	Workable rule but not in terms of size number
960	$n + 3$
970	Incomplete or just describe how a particular case was obtained
980	Incorrect
990	Blank

APPENDIX 6(B): CODING SCHEME FOR RULE TYPES FOR OH DEER!

SPSS Code	Rule generated for Q2 (Oh Deer!)
201	$n^2 + 3n + 2$
202	$(n + 1)(n + 2)$
203	$n(n + 1) + 2(n + 1)$ $n(n + 1) + 2n + 2$
204	$n^2 + 2n + (n + 2)$
205	$n^2 + n + 2(n + 1)$
206	$n(n + 3) + 2$
207	$n(n + 2) + (n + 2)$
208	$(n + 1)^2 + (n + 1)$
209	$n^2 + 8 + 3(n - 2)$
220	Recursive rule showing how the next term is obtained from the previous term (the first and second differences are identified) (e.g., $T_n = T_{n-1} + 2(n + 1)$, $T_{n+1} = T_n + 2(n + 2)$, or for the n^{th} term, add $2(n + 1)$ to the previous term.)
221	First difference is identified only (eg, the pattern is + 6, +8, +10, +12, …etc or add multiples of two)
950	Workable rule but not in terms of size number
960	$n + (multiples\ of\ two)$
970	Incomplete or just describe how a particular case was obtained
980	Incorrect
990	Blank

APPENDIX 6(C): CODING SCHEME FOR RULE TYPES FOR BIRTHDAY PARTY DECORATIONS

SPSS Code	Rule generated for Q3 (Birthday Party Decorations)
301	$3n + 2$
302	$5 + 3(n - 1)$
303	$2(n + 1) + n$
304	$3(n + 2) - 4$
305	$3(n + 1) - 1$
306	$2n + (n + 2)$
307	$n + n + (n + 2)$
320	Recursive rule showing how the next term is obtained from the previous term (e.g., “Add 3” rule, increase the cards by 3, or add 1 card to each column)
950	Workable rule but not in terms of size number
960	$n + 3$
970	Incomplete or just describe how a particular case was obtained
980	Incorrect
990	Blank

APPENDIX 6(D): CODING SCHEME FOR RULE TYPES FOR TULIPS

SPSS Code	Rule generated for Q4 (Tulips)
401	$n^2 + 2n$
402	$n(n+2)$
403	$n + n(n+1)$ $n + (n^2 + n)$ $n + \frac{2n(n+1)}{2}$
404	$n + 2[n + (n-1) + (n-2) + \cdots + 3 + 2 + 1]$
405	$3n + n(n-1)$
406	$(n+1)^2 - 1$
407	$(2n+1)(n+1) - n(n+1) - 1$
408	$n(2n+1) - n(n-1)$
409	$8 + (n+4)(n-2)$
410	$(n^2 - 1) + (n+1) + n$
420	Recursive rule showing how the next term is obtained from the previous term (the first and second differences are identified) (e.g., $T_n = T_{n-1} + (2n + 1)$, $T_{n+1} = T_n + (2n + 3)$, or for the n^{th} term, add $(2n + 1)$ to the previous term)
421	First difference is identified only (eg, the pattern is + 5, +7, +9, +11, …etc or add odd numbers)
950	Workable rule but not in terms of size number
960	$n + (odd\ numbers)$
970	Incomplete or just describe how a particular case was obtained
980	Incorrect
990	Blank

APPENDIX 6(E): CODING SCHEME FOR RULE TYPES FOR CHRISTMAS PARTY DECORATIONS

SPSS Code	Rule generated for Q5 (Christmas Party Decorations)
501	$n^2 + 2n + 2$
502	$n(n+2) + 2$ $n(3 + n - 1) + 2$
503	$n^2 + 2(n+1)$
504	$(n+1)(n+2) - n$
505	$n(n+1) + n + 2$
506	$(n+1)^2 + 1$
507	$n(n-1) + 3n + 2$
508	$(n+1)^2 - 2(n+1)$
520	Recursive rule showing how the next term is obtained from the previous term (the first and second differences are identified) (e.g., $T_n = T_{n-1} + 2n + 1$, $T_{n+1} = T_n + 2n + 3$, or for the n^{th} term, add $2n + 1$ to the previous term)
521	Listing the first differences only (eg, the pattern is +5, +7, +9, +11, …etc or add odd numbers)
950	Workable rule but not in terms of size number
960	$n + (odd\ numbers\ 5, 7, 9, 11, \ldots)$
970	Incomplete or just describe how a particular case was obtained
980	Incorrect
990	Blank

APPENDIX 6(F): CODING SCHEME FOR RULE TYPES FOR TOWERS

SPSS Code	Rule generated for Q6 (Towers)
601	$4n + 2$
602	$6 + 4(n - 1)$
603	$2(n + 1) + 2n$
604	$2n + (n + 2) + n$
605	$(n + 1)(n + 2) - n^2 + n$
606	$(n + 1)(n + 2) - n(n - 1)$
607	$10 + 4(n - 2)$
608	$2(2n + 1)$
609	$2n + 2n + 2$ or $2(2n) + 1$
610	$2(n + 2) + 2(n - 1)$
611	$(n + 2)^2 - n^2 - 2$
612	$5n - (n - 2)$
613	$2n + 2(n - 1) + 4$
614	$3n + (n + 2)$
615	$6n - 2(n - 1)$
616	$4\left(n + \frac{1}{2}\right)$
620	Recursive rule showing how the next term is obtained from the previous term (e.g., "Add 4" rule, increase the tiles by 4, or add 1 tile to the left and to the right column and 2 tiles to the middle block)
950	Workable rule but not in terms of size number
960	$n + 4$
970	Incomplete or just describe how a particular case was obtained
980	Incorrect
990	Blank

APPENDIX 6(G): CODING SCHEME FOR RULE TYPES FOR WALL DESIGN

SPSS Code	Rule generated for Q7 (Wall Design)
701	$n^2 + 2n + 1$ $n^2 + (2n + 1)$
702	$(n + 1)^2$
703	$n(n + 2) + 1$
704	$n(n + 1) + (n + 1)$
705	$4n + (n - 1)^2$
706	$(2n + 1)(n + 1) - n(n + 1)$
707	$n(n - 1) + 3n + 1$
720	Recursive rule showing how the next term is obtained from the previous term (the first and second differences are identified) (e.g., $T_n = T_{n-1} + 2n + 1$, $T_{n+1} = T_n + 2n + 3$, or for the n^{th} term, add $2n$ + 3 to the previous term)
721	Listing the first differences only (eg, the pattern is + 5, +7, +9, +11, …etc or add odd numbers)
950	Workable rule but not in terms of size number
960	$n + (odd\ numbers\ 5, 7, 9, 11, \dots)$
970	Incomplete or just describe how a particular case was obtained
980	Incorrect
990	Blank

APPENDIX 6(H): CODING SCHEME FOR RULE TYPES FOR HIGH CHAIRS

SPSS Code	Rule generated for Q8 (High Chairs)
801	$3n + 5$
802	$8 + 3(n - 1)$
803	$2(n + 1) + (n + 3)$
804	$3(n + 1) + 2$
805	$3(n + 2) - 1$
806	$3(n + 3) - 4$
807	$11 + 3(n - 2)$
808	$2(n + 3) + (n - 1)$
809	$6 + (3n - 1)$
810	$5(n + 1) - 2n$
811	$2(n + 2) + (n + 1)$
812	$(3n - 1) + 6$
813	$4n + 4 - (n - 1)$
820	Recursive rule showing how the next term is obtained from the previous term (e.g., “Add 3” rule, increase the cards by 3, or add 1 card to the left, middle and right column)
950	Workable rule but not in terms of size number
960	$n + 3$
970	Incomplete or just describe how a particular case was obtained
980	Incorrect
990	Blank

APPENDIX 7: CODING SCHEME FOR THE MODALITY OF RULES

SPSS Code	Descriptor
1	Completely in words Eg, size add one and multiply it to size add two
2	Completely in notations Eg, $(n+1)(n+2)$
3	In alphanumeric form Eg, $(size\ number+1)(size\ number+2)$
4	Incomplete rule or description of how a particular figure was obtained
5	Incorrect, irrelevant rule
9	Blank

APPENDIX 8: CODING SCHEME FOR THE GENERALISING STRATEGIES

SPSS Code	Generalising Strategy
11	Find difference only, leading usually to the recursive rule
12	Repeated substitution: express terms in terms of immediate term preceding it (inductive approach)
13	Comparison: compare terms with sequence whose rule is known
14	Substitute values into formula: (1) p = difference, q = zeroth term into $pn + q$ (2) a = first term, d = difference into $a + (n - 1)d$
15	Find difference, formulate and solve equations
16	Find difference, compare and equate expressions
17	Find number of groups
21	Constructive
22	Deconstructive
23	Reconstructive
24	Figure-ground reversal
2113	Constructive followed by comparing terms with sequence whose rule is known
2123	Constructive followed by reconstructive
2124	Constructive followed by figure-ground reversal
2321	Reconstructive followed by constructive
2324	Reconstructive followed by figure-ground reversal
2423	Figure-ground reversal followed by reconstructive
31	Guess-and-check or pattern spotting (shows numerical structure leading to the rule, NOT *verifying* rule)
32	Look at nature of terms
41	Correct rule using an indeterminate strategy
42	Response is partially complete
43	Incorrect strategy
44	A particular case was described
99	Blank

APPENDIX 9: CODING SCHEME FOR THE JUSTIFICATION SCHEMES

Category	Code	Description
justify explicit recursive rule	11	Indicate the differences between consecutive terms listed as **a sequence**
	12	Indicate the differences between consecutive **configurations**
	13	Describe how the difference was obtained
	14	Organise the terms **in a tabular form**, indicating the difference
justify explicit functional rule **without** diagrams	21	(a) Substitute a few values into rule to verify its correctness (b) Provide diagrams of the configurations to verify its correctness (***justification focuses on ascertaining output values rather than on the components***)
	22	Organise numerical structures of configurations **in a tabular form** to show how the rule was obtained
	23	Provide **a few numerical structures** of configurations to show how the rule was obtained
	24	Provide **a generic structure** of a configuration to show how the rule was obtained
	25	Describe in words the steps for obtaining the rule by comparing with a known sequence (usually if generalising strategy code 13 is used)
	26	Describe in words the steps for obtaining the rule by substituting values into formula (usually if generalising strategy code 14 is used)
	28	Provide working to show how the rule was obtained by solving equations (usually if generalising strategy code 15 or 16 is used)
	29	Demonstrate evidence of guess and check
justify explicit functional rule **with** diagrams	31	Provide **a few** configurations and elaborate how the rule was obtained
	32	Provide **a generic** configuration and elaborate how the rule was obtained
	33	Provide **a few** configurations and organise values **in a tabular form** to show how the rule was obtained
justify rule not in any of the categories above	41	Provide examples in words or diagrams as illustrations to elaborate the rule not linked to size number
	42	Provide examples in words or diagrams as illustrations to elaborate the rule linked to size number

miscellaneous	94	Provide diagrams of configurations only
	95	Repeat the rule, either in words, symbols or a combination of both (eg, $3n + 2$ is obtained, but justification shows *n times size no, then add 2*)
	96	Provide a justification related to the given rule but never describe how the rule was obtained (eg, Tulips: from the total tiles and diagrams)
	97	Provide a wrong or irrelevant justification (eg, use rule, they increase in size number)
	98	Provide no justification for rule
	99	Blank